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ABSTRACT

The aim of this study was to identify deficiencies in students' knowledge and understanding of basic algebraic concepts and operations. Since analysis of errors is an effective way of tracing misconceptions in mathematics, a diagnostic test was compiled for this purpose. A total of 162 items was included in the test which was administered to 4,635 standard nine students from 106 schools in the Republic of South Africa. Although most students were supposed to find a diagnostic test easy, it was discovered that many of the students experienced serious difficulties. The main cause of the difficulties was attributed to a lack of understanding. The students often succeeded in some manipulations but failed to understand the concepts and principles on which these manipulations were based. It was concluded that deficiencies in basic prerequisites included ignorance of algorithms, inadequate mastery of basic facts, incorrect procedures in applying mathematical techniques, and insufficient knowledge of important concepts and symbols. A total of 53 references are listed. (Author/YP)

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Poor concept formation in Mathematics: a diagnostic perspective

J.J. Barnard

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Poor concept formation in Mathematics: a diagnostic perspective

Report P-105

Poor concept formation in Mathematics: a diagnostic perspective

J.J. Barnard

Pretoria
Human Sciences Research Council
1989

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PREFACE

Mathematics plays an important role in the development of modern society. The highly technicalized and industrialized achievement–oriented society depends to a large extent on the maintenance of high quality Mathematics instruction for upholding the sophisticated lifestyle of its people. Many occupations require a certain degree of mathematical knowledge and skill and consequently Mathematics will also help the non–mathematician, the applier of mathematical knowledge, to do his work more effectively and with greater insight.

Mathematics is generally considered a "difficult subject". This reputation can perhaps be attributed partially to the fact that from their first school years pupils do not adequately master certain fundamental mathematical concepts and principles (which are essential for academic progress), with the result that the subject contents become increasingly incomprehensible to them. Several fundamental questions are left unanswered and the child's subsequent failures increasingly reflect his/her inability to deal with mathematical contents with understanding. He/she is therefore forced to fall back on memorization and "recipe" methods.

The research on which this report is based was undertaken in order to gain an idea of the extent and nature of mistakes made by pupils in Algebra which can be attributed to the superficial and/or incomplete formation of basic concepts. The results clearly revealed that this was in fact a fundamental cause of poor performance in Mathematics and it is believed that similar research results would also be obtained in the case of Geometry and Trigonometry.

R. J. Prinsloo

EXECUTIVE DIRECTOR

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EKSERP/AβSTRACT

Aanspraak kan nie daarop gemaak word dat basiese begrippe en beginsels die enigste faktore is wat prestasie in Wiskunde beïnvloed nie. Aangesien Wiskunde egter by uitnemendheid 'n hiërargiese struktuur het, kan die aanspraak wel gemaak word dat indien basiese begrippe onvolledig en/of oppervlakkig in die kognitiewe struktuur gevorm is, 'n onvermoë om te vorder toenemend duideliker sal word. 'n Moontlik aanvanklike skynsukses weerspieël spoedig in wanprestasie.

Die doel met hierdie studie was om gebreke in leerlinge se kennis oor en begrip van basiese algebraïese konsepte en bewerkings te identifiseer. Aangesien foute-ontleding 'n doeltreffende wyse is waarop wanbegrippe in Wiskunde opgespoor kan word, is 'n diagnostiese toets vir hierdie doel saamgestel. 'n Totaal van 162 items is in die toets ingesluit wat deur 4 635 standerd nege-leerlinge uit 106 skole in die Republiek van Suid-Afrika beantwoord is. Alhoewel 'n diagnostiese toets vir die meeste leerlinge maklik behoort te wees, is gevind dat baie leerlinge ernstige probleme ondervind het. Die primêre oorsaak skyn te wees dat leerlinge nie verstaan wat hulle doen nie. Daar is dikwels korrekte antwoorde op berekeninge gegee, maar sodra insig in die onderliggende begrippe ter sprake kom, faal baie leerlinge. Daar is tot die gevolgtrekking gekom dat algoritmes dikwels toegepas, formules gebruik en reëls slaafs nagevolg kan word, maar dat die begrippe en beginsels waarop dit gebaseer is, nie werklik bemeester is nie.

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It would be presumptuous to claim that fundamental principles and basic concepts are the only factors influencing achievement in mathematics, but since mathematics has a hierarchical structure, an understanding of the basic concepts and principles of mathematics will facilitate the acquisition and retention of mathematical skills. If a particular level is imperfectly understood, more advanced levels become increasingly out of reach and progress from then on is problematic.

The aim of this study was to identify deficiencies in pupils' knowledge and understanding of basic algebraic concepts and operations. Since analysis of errors is an effective way of tracing misconceptions in Mathematics, a diagnostic test was compiled for this purpose. A total of 162 items was included in the test which was administered to 4 635 Standard nine pupils from 106 schools in the Republic of South Africa. Although most pupils are supposed to find a diagnostic test easy, it was found that many of the pupils experienced serious difficulties. The main cause of most of the difficulties can be attributed to a lack of understanding. The pupils often succeeded in some manipulations but failed to understand the concepts and principles on which these manipulations were based. It was concluded that deficiencies in basic prerequisites included ignorance of algorithms, inadequate mastery of basic facts, incorrect procedures in applying mathematical techniques and insufficient knowledge of important concepts and symbols.

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CHAPTER 1
GENERAL ORIENTATION

1.1 INTRODUCTION

Renewal in teaching has become a buzz word. Since the turn of the century there has been a shift in emphasis towards the child in the teaching–learning situation. Whereas children were originally seen as empty vessels waiting to be filled with knowledge by adults, they came to be regarded as people with creative potential which had to be realized. An attempt was made by means of programmed teaching, team teaching, computer supported and computer based education, audiovisual aids and so on to refine increasingly the execution of the teaching events. Attempts were made to modernize teaching in respect of contents and also of objectives and methods. Individualization, understanding and insight are among the terms which became very prominent. Since the New Maths appeared on the scene in the nineteen–sixties, meaningfulness as a prerequisite for the learning process has been emphasized. Meaning and structure have come to be of prime importance. These changes undergone by traditional schools have been called the "Third Great Revolution of Mankind" by Frankel, while Drucker refers to "The Educational Revolution" (Steenkamp, 1980:1).

The new approach to education has also come to the fore in the teaching of Mathematics and the natural sciences. The launching of Sputnik I by the Russians in 1957 compelled the Western world to give serious consideration to its teaching methods. Conventional methods and conformist attitudes in schools were seen as factors contributing to the loss of the space race (Strauss, 1982:8).

Any Mathematics teaching which involves more than mere memorization has got to take the formation of concepts into account. Mathematics has complex contextual rules and the mastering of a mathematical vocabulary can be regarded as an indirect mastering of these rules and consequently of the concepts required for an understanding of Mathematics. This mathematical vocabulary is also a component of the language of teaching and as such it becomes the way in which transfers of knowledge take place. Pupils who do not understand the vocabulary will not benefit from the teaching. In the words of De Wet (1961:50/51): "Language is very important in the formation of concepts because the pupils use it to increase the

effectiveness with which they acquire concepts. We therefore want to draw attention to the relationship between language and the formation of concepts. Words are symbols for concepts and have three relationships with them:

1. The syntactic dimension. Words and symbols derive their meaning from the context in which they are used. The root of an equation and the root of a tree are two separate concepts with distinct meanings.

2. The pragmatic dimension. Words and symbols change according to their use, their aims and consequences.

3. The semantic dimension. This is the most important to mathematicians. Words have meaning according to their relationship with the objects they refer to. Children learn what a chair is after repeatedly hearing the word applied to a four-legged thing that people sit on. Language also plays an important part in the formation of mathematical concepts. It is the medium through which teachers convey their own concepts to the children. ... If the language of Mathematics is understood, it will promote the pupils' formation of concepts and their mathematical knowledge. The contrary also applies. If pupils do not understand the language of Mathematics, they will be at a disadvantage in every attempt to form mathematical concepts or to expand their mathematical knowledge." Language and concepts can therefore be regarded as correlates of each other, which is why pupils have to master the symbolic language of Mathematics in order to establish a world of meanings of their own.

A minimal amount of ready knowledge and a specific level of skill at mathematical manipulation form the basis of all mathematical operations. These indispensable basic concepts and principles are defectively mastered from the start, which makes the subject progressively more unintelligible to children. Many fundamental questions are left unanswered and the mishaps which result from this increasingly reflect the children's inability to work insightfully with mathematical topics. They are forced to use recipe methods and memorization.

It is generally accepted that progress in Mathematics depends on the mastery of fundamental mathematical concepts and principles. De Wet (1966:111/112) refers to Sinnema, whose investigation found that "the lack of a rational basis for simple arithmetical operations is one of the major causes of poor achievement in mathematics". De Wet also refers to Van Rooy, who regards poor grounding in Algebra as a significant cause of the defects in the mathematical surface structure of first-year students of Mathematics. Similarly, Du Toit (1975:1) says: "The problem

of underachievement and the deleterious consequences it has for society at present is a topic which is enjoying increasing attention abroad and in our country. Failures at the secondary school level can frequently be traced back to underachievement in the primary school, so it is essential to be able to identify these cases early". Gouws (1968:1) poses the question: "Why do employers complain that pupils who leave school after Std 8 have not mastered even the basic principles of the subject?"

Didacticians must view this in a very serious light, because the role of Mathematics in society is growing in importance. Mathematics is a basic component of all scientific and technical research and training. Our achievement–oriented, technicized and industrialized society is actually a hostage to the maintenance of high–quality Mathematics teaching for the preservation of its people's existing sophisticated lifestyle (cf. Barnard, 1985:2). For its continuation and maintenance, modern scientific, technological, industrial and computerized society depends on effective and responsible Mathematics teaching in institutionalized education.

1.2 STATEMENT OF THE PROBLEM

A characteristic feature of Mathematics is the hierarchical building up of concepts. Should one of these components be lacking or poorly developed, the result may be a defective ability to think mathematically. If a specific level is imperfectly understood, more advanced levels become ever more inaccessible and progress will be problematical.

Subjective experience acquired in the field of teaching has revealed that pupils often lose marks in formal examinations in Mathematics because of the cumulative effect of a lack of knowledge of basic mathematical concepts and their inability to handle these concepts to good effect. In the light of the foregoing, a comprehensive empirical study was undertaken to determine the extent and nature of the inadequate knowledge of basic mathematical concepts and principles and its effect on pupils' achievement in Mathematics. Standard Nine pupils who take Mathematics as a subject were included in the study because these pupils have chosen Mathematics as one of their subjects and have also made progress with the concepts of the senior secondary phase. Problems with underlying concepts of the previous standards will consequently become clear at this stage.

1.3 AIM OF THE RESEARCH

The basic assumption is that at any school level there are certain fundamental principles, knowledge and skills which have to be mastered by all the pupils. This basic reference framework which is built on and without which progress cannot be made is of particular importance in Mathematics. The primary aim of the study is to determine to which extent the pupils have mastered the basic knowledge, principles and skills which underlie the sul ject contents of Mathematics. Secondly, the nature and extent of specific problems will be determined. The investigation thus involves the measurement of basic knowledge and the execution of simple calculations and manipulations in Algebra.

1.4 DEFINITIONS OF CONCEPTS

In this section some frequently-used concepts will be defined briefly.

An *item* is the smallest single unit of a test, it is the single question together with its possible answers. An item consists of a stem — the section of the item posing the question — and choices which include the correct answer as well as distractors. An *equivalent item* is an item identical to the original item except that figures are replaced by other figures and/or letters by other letters.

Achievement refers to the degree to which pupils have at their disposal the knowledge, skills, intelligence and other abilities required for selecting the correct answers to items. The higher a pupil's achievement, the more items have been answered correctly.

Where the score for an item can be 0 or 1 the average score is the simplest index which gives an indication of how difficult the item is. This index is known as the *difficulty value* of the item. The difficulty value of a dichotomous item is defined (see for instance Van den Berg and Vorster, 1982:79 or Huysamen, 1980:123) as the ratio of candidates who answered the item correctly to all the candidates, that is

$$m_i = \frac{N_r}{N_t}$$

where

m_i = difficulty value of item i,

N_r = number of candidates that got the correct answer and

N_t = number of candidates in the sample.

The higher the value of the ratio, the easier the item. In this study the ratio is expressed as a percentage so that the difficulty value of an item is taken as the percentage of pupils who answered the item correctly.

The *discrimination value* of an item is the product moment correlation between the item score and the total score of the test as internal criterion (see for instance Van den Berg and Vorster, 1982:81). This index refers in other words to how well the item distinguishes between candidates who generally did well and those who generally did badly. The higher the index, the better the discrimination. For dichotomous items, the point biserial correlation between the item and the total score for the test is calculated to give an indication of the discrimination value. The following formula (Van den Berg and Vorster, 1982:81) was used:

$$d = \frac{X_r - X_t}{S_t}\sqrt{\frac{p}{q}}$$

where

d = point biserial correlation coefficient,

X_r = average test score of pupils who got the item right,

X_t = average test score of the whole group,

S_t = standard deviation of the test scores of the whole group,

p = proportion of pupils who got the item right and

q = proportion of pupils who got the item wrong $(1 - p)$.

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CHAPTER 2
FUNDAMENTAL ROLE OF CONCEPTS IN MATHEMATICS

2.1 INTRODUCTION

Teachers of Mathematics are constantly being confronted by concepts like *function*, *variable*, *factor*, *set* and many others, the meaning of which must be communicated to the pupils. Mathematics is not merely a mastering of calculations and memorizing of formulas, but largely consists in giving meaning to complex, abstract concepts. Concepts initially formed as suggested by the environment are then generalized to a high degree of abstraction. Strauss (Malan, 1980:39) regards Mathematics as a structure of mathematical concepts which are extended in a spiral so that the learner finds himself on a higher level every time a cycle is completed.

2.2 CONCEPTS

Teachers of Mathematics must be informed about the ways in which mathematical concepts are formed by children. Before discussing this, we must first be sure about what is meant by concepts. The rich literature on and around this topic does not provide unanimity about what exactly a concept is. (Cf. also Austin and Howson, 1979:167.) The following exemplary definitions illustrate this point:

Spitzer (1977:23/24) sees concepts as abstract aids which man develops in order to control the complexity of the real world. They are groups of "things" to which common names are given. The author distinguishes further between external and internal concepts. External concepts are concepts developed jointly by people, while internal concepts are developed by individuals from their own experiences. Cronjé (1957:41/42) largely agrees with this when distinguishing between experiential concepts and logical/abstract concepts.

Behr (1974:27) sees concepts as ideas. The formation of such an idea is always preceded by experiences which have something in common. "Concepts have their origin in sensory experience but depend upon mental structures, called schemas, for their formation." These schemas are not static since new information is constantly being assimilated and new situations and experiences are being accommodated. Natural numbers are thus one of the first basic mathematical schemas which children learn. After assimilation one can judge other objects, ideas and so on and decide whether or not they are examples of a specific concept.

Cooney and Davis (1976:216 et seq.) distinguish two components of a concept, a demonstrative expression and a referent. The referent may be a set of objects or characteristics associated with a collection of objects. In Mathematics the concept of **rhombus** consists for instance of the term (name) *rhombus* together with the referent ***four sided equilateral figure***. The referent of a concept like **parallel** is a number of characteristics rather than a collection of objects.

Van den Berg (1980:4) defines concept formation as complex thought transactions which include the differentiating,classifying, organizing, generalizing and abstracting of objects and experiences. He emphasizes the cardinal importance of one's own experiences so that one forms concepts for oneself from one's own experience. The direct implication of this is that children must be intellectually and emotionally mature before they can acquire higher level concepts. If children are forced to learn concepts before they are ready, then at best they will know them but they will not understand them.

Kathan (1969:1 et seq.) made a comprehensive theoretical study of concept formation. After listing numerous definitions from general, philosophical and logical, psychological and other specialized sources, he arrived at the following definition: "A concept is a mental form which results from common features universally accepted which have been isolated from objects or events. Because of this the individual is able to put events or objects which possess these universally accepted characteristics (similarities) into a category (class). The characteristics of a group of objects or events may be expressed verbally or otherwise. This concept which has been formed could be applied in a novel situation".

Klausmeier, Ghatala and Frayer (1974:4/5) define a concept as organized information about the characteristics of things (objects, events, transactions) in terms of which any thing or class of things can be distinguished. These authors identify eight features of concepts:

a. **Ease of learning**: All types of concepts are not equally easy or difficult to grasp. Abstract concepts are much harder to acquire than concrete concepts.

b. **Utility**: Certain concepts are used more than others.

c. **Validity**: Experts agree more about certain concepts than others. The concept *intelligence* is a good example of a concept for which psychologists will probably never agree on a definition.

d. **Universality**: The more specific a concept is, the more features are used to describe it. *Animal* has fewer defining features than ***kitten***. The higher a concept is situated in a taxonomy, the more general it is in terms of the concepts it includes.

e. **Force**: The force of a concept refers to the extent to which it facilitates the learning of another concept. The more vital it is to understand a concept in order to learn another one, the more force it is said to have.

f. **Structure**: A concept described in terms of features possesses a structure defining the relations between the features.

g. **Observability**: Some concepts like *tree* can be directly observed while others like ***infinity*** are not directly observable.

h. **Numerousness**: The number of components of a concept can vary. *Moon* has only one referent with respect to the earth, while ***number*** has infinitely many (1, $\frac{2}{3}$, ...).

There are many theories about the formation of (mathematical) concepts (cf. Barnard, 1988:51 et seq.). Although the theories may seem divergent, there are many common elements which have important didactic implications. These common elements are put into didactic context in the following section by isolating five principles for the teaching of mathematical concepts. (No claim is made that this set of principles is complete.)

2.3 THE TEACHING OF CONCEPTS AND PRINCIPLES

2.3.1 Pupils themselves form concepts

Teachers cannot give their pupils a concept. All pupils have to formulate concepts for themselves. Teachers' prime responsibility is to create learning experiences which make it possible for the pupils to master the concepts. In the primary school phase especially the availability of concrete material is vital. Children who have to master the concept *fraction* via pictures in a textbook will be more likely to form a vague concept than children who have handled concrete materials themselves. Teachers often expect their pupils to visualize the problem set for them but do not ensure that the pupils have a firm concrete background on which to build.

2.3.2 Abstracting must not be enforced

Lower order concepts must be properly grasped by means of concrete experiences before pupils can advance to higher order concepts (abstract level). The practice of Mathematics with insight and the handling of abstract concepts must be preceded by control of fundamental principles and concepts. If pupils are confronted too soon by abstract principles which they are not yet ready for, they may be forced into instrumentalism (De Villiers, 1985:108). These pupils will in other words regard Mathematics as a collection of rules, procedures and algorithms which have to be memorized in order to solve problems. For instance we cannot simply move on from arithmetical constants to variables and algebraic fractions. Algebraic fractions can not be meaningfully taught if the pupils have only a vague understanding of rational numbers. The use of symbols, especially for abstract concepts, must be preceded by a real understanding of the meaning of the concept. The symbol + should for instance not be used until the concept *addition* has been grasped and named.

An appearance of success from an initial "good beginning" will soon lead to an inability to make progress if pupils do not catch on to underlying principles before moving to the abstract level.

2.3.3 Intellectual and emotional maturation

Experts agree that pupils should be intellectually and emotionally mature in order to master concepts. As Cronjé (1957:6) puts it: "The breakthrough of insight obtained from learning-thinking must not be forced but is dependent on an internal maturation process in the child". Copeland (1970:23) agrees that " ... there is a readiness stage that the child must reach before logical concepts can be taught

or rather learned". One should wait for the (physical and biological) process of neurological maturation and children should not be forced too quickly through a Mathematics programme. If the concepts are forced on to pupils before they are ready, the best that they can do is to memorize them. They will not be able to give any meaning to the concepts and only parrot–like behaviour will be possible. Bosch (1969:155/156) describes teaching before pupils are ready for it as dressage. Children must have the necessary prior knowledge, skills and experiences to be able to acquire a new concept. While this prior knowledge is being actualized, the teacher must meet the pupils where they are and not where s/he thinks they should be.

2.3.4 Give sufficient examples

Pupils cannot master a concept with the aid of just one example or even several. The concept must be brought home to them through numerous positive and negative examples. If teachers provide enough examples for the pupils, they will be able to form the concept. Pupils must be able to distinguish the essential from the non–essential through the examples. Let us say that pupils have to master the concept *triangle*. All triangles are part of the set of all geometrical figures. In order to decide whether a given figure is a triangle the number of sides of the figure must be taken into account while dimensions such as the lengths of the sides, sizes of the angles and so on are irrelevant. The teacher's positive examples will thus include triangles of various sizes and with different angles; rectangles, rhombuses and circles may be used as negative examples. It is obvious that if circles for instance have been introduced before triangles, fewer examples will be needed before pupils can distinguish between triangles and circles than when they have to distinguish between for instance circles and ellipses. The examples should illustrate the maximum number of features and teachers should point out the connections between the various examples which will give rise to the identification of the characteristic features of a concept. It is also important that pupils should be able to associate concepts of the same class. In this way generalization and the formation of higher order concepts will be able to take place meaningfully.

2.3.5 Early mechanization is undesirable

The author suspects that many misconceptions are traceable to a tendency among teachers to teach the pupils "shortcuts" before concepts have been properly mastered. How often pupils talk about "borrowing" when they subtract, "moving the

point" when they work with decimals, "cancelling" and so on. The consequences of this show clearly in their meaningless manipulation of symbols.

Many pupils think of cancelling as "crossing out". The process is not associated in any way with a mathematical law and this leads to problems for pupils. They cannot understand why $\frac{2}{3} \times \frac{7}{2} = \frac{7}{3}$ while $\frac{2}{3} + \frac{7}{2} \neq \frac{7}{3}$. Because the underlying principles have not been properly grasped, the pupils cannot see why $\frac{3(a+b)}{3(m+n)} = \frac{a+b}{m+n}$ while this does not apply to $\frac{3a+b}{3m+n}$. The pupils probably do not understand the difference between the concepts *factor* and *term*. Pupils who know that $\frac{x}{y}$ means "x of the $\frac{1}{y}$ fractions" will not try to add $\frac{x}{y}$ and $\frac{a}{b}$ or to write $\frac{2+a}{x+y}$ as $\frac{2}{x} + \frac{a}{y}$.

It is vital that pupils should first be able to identify the principle or law on which an operation (in this case simplification) is based and also that it should be meaningful to them before the process is mechanized. For solving an equation like $x + 2 = 5$, pupils are often taught to reason as follows: move the 2 across the =, change its sign and add it to the 5. This does give the correct answer, but if the pupils do not know that the reason why it can be done that way is based on the principle of equal sections of equal things leaving equal remainders, they will never really know why the correct answer can be obtained in this way. Cross multiplication is usually learned as an algorithm without reference to the identity element of multiplication or to the fact that if both sides of an equation are multiplied by equal numbers, the results are equal.

Teachers of Mathematics must be able to account for themselves in answer to such questions as:

Are pupils given enough time for considering, processing and assimilating new concepts?

Are concepts that have been mastered related to each other and to the total structure of which they form a part?

Are new concepts taught by means of out–of–context definitions instead of being connected to the children's frame of reference?

Is a concept taught by means of an algorithm?

Are enough positive and negative examples given and are they sufficiently varied?

Are the pupils regarded as passive recipients of knowledge with little opportunity for creative participation in activities which allow them to construct concepts for themselves?

Are pupils overwhelmed too quickly with details and special cases — even before the concept has been properly grasped?

Is Mathematics taught as a closed system based on the teacher's authority?

Is mechanization encouraged for the sake of getting answers but at the cost of revising fundamental underlying mathematical laws and principles?

Are syllabuses rushed through because of pressure of time?

Is the movement from concrete to abstract made too soon?

What is clear from the above is the fact that Mathematics should not be taught as an end product. Children who are taught to accept everything on their teacher's authority and not to think for themselves will experience problems when they are exposed to new challenges. Pupils must learn to reason critically and to question statements. Barrow (1981:45) has this to say on the subject: "One might reasonably say that one clear goal of education is developing powers of critical thought. For what I mean by critical thinking is thinking that is concerned about and embodies good reasoning and coherent steps, conceptual clarity and discrimination in planning, discussion, explanation and any other form of ratiocination." Teachers who see their job as merely getting correct facts across will give rise to the memorization data without insight and with the exclusive aim of finding answers. This method of teaching suppresses the creative potential of the learning child.

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CHAPTER 3
METHOD OF INVESTIGATION

3.1 INTRODUCTION

Because diagnostic tests have as their objective a systematic explication of the nature and extent of specific problems in respect of a particular subject area, a diagnostic test was developed as a measuring instrument to obtain interpretable data on the level of achievement and the nature and extent of errors of the testees. The test is based on the assumption that problems that pupils have in the handling of fundamental mathematical concepts and principles can be traced by noting which distractor a pupil chooses in a multiple choice item. An analysis of the pupils' responses should point to specific problems which they are experiencing in this regard. We also work on the assumption that a pupil's ability to handle fundamental mathematical concepts and principles is a valid criterion for his/her level of achievement in Mathematics in general.

The investigation was conducted in two phases namely a pilot study with a small group of Standard Nine pupils and a final application with a relatively big sample of Standard Nine pupils.

The results (in both applications) of the test are divided into three categories of which each are divided into a number of groups. The categories are

— the use of fundamental concepts, principles and skills,

— mathematization and

— demathematization.

The first category includes sixteen groups (topics) relating to basic concepts, principles and skills. The sixteen groups are priority of operations, graphs, inequalities, substitution, roots, factorization, formulas, decimal numbers, impossible problems, principle operations, exponents, fractions, solving equations, percentages, ratios, and unnecessary information. A pupil who cannot make substitutions in formulas, for instance, is lacking in an essential component for the practice of Mathematics. The second category includes the processes of translation from ordinary language to the language of Mathematics. Typical items in this category will include verbal problems as well as items about communicative language and technical language. Items in this category are divided into four groups, namely notation, terminology, colloquial language and conversions. As has been said several times already, the formation of concepts cannot take place without language. The aim of the inclusion of the language aspects of the second category should not be seen as a complete study but rather as a supplement to the first and an indication of general tendencies. Demathematization involves the interpretation and attribution of meaning to the answers or solutions obtained for problems.

Items with five choices were written for the purpose of the study after a study of relevant literature was undertaken. Five choices were considered adequate for including the majority of pupils' answers, although there are always pupils whose errors will be unclassifiable. An attempt was made throughout to keep the items simple, homogenous, unambiguous and emotionally neutral and to ensure that they demanded little reading and writing from the pupils. The items included topics from the syllabuses of standard two up to standard eight. (The type of items will be discussed in the next chapter.) Before being used the items were presented to an internal committee of the Institute for Psychological and Edumetric Research of the HSRC for recommendations and amendments.

3.2 THE PILOT STUDY

The final application was preceded by a pilot study in which 61 Standard Nine pupils at one Afrikaans–medium secondary school (which was not included in the sample) were included. All pupils taking Mathematics as a subject wrote the test (40 pupils took the subject on the higher grade, 19 on standard grade and two on lower grade).

The test consisted of 237 items divided into five subtests. The different types of items were distributed at random through the subtests and the pupils were

encouraged to answer every item. The pupils had a break after completing the first two subtests and after completing subtests three and four.

The tests were marked by hand and an item analysis was carried out. Item analysis involves the statistical processing of as many relevant facts as possible about the items of a test. The objectives of item analysis are summarized as follows by Heinichen (Malan, 1981:22):

i." To obtain objective information regarding the written item with which the subjective judgement of the test setter about the choice of such an item may be checked.

ii. To collect such information in connection with items that the test which will eventually be compiled from them will have specific features in respect of degree of difficulty, reliability and validity.

iii. To discover defects in items which can be corrected so that poor items can be made relatively good.

iv. To be able to determine the most suitable items for a test."

The data obtained was processed by means of a computer. Although the difficulty and discrimination values of each item were calculated, when selecting items for the final application it was borne in mind that the test is a diagnostic one and that all the items should have been answered correctly by all the pupils. Items which did not meet the required conditions were eliminated or improved if they had been included for a specific reason. The distractors of some of the items were replaced if the response frequencies were too low. This was an attempt to ensure that suitable items were objectively selected for the final application. A total of 162 items were regarded as suitable for the investigation. These items were translated into English and once more submitted to the internal committee for comment, discussion and amendment.

3.3 THE FINAL APPLICATION

3.3.1 The sample

A stratified random sample with the following subdivisions was taken from the population of all Standard Nine pupils in the RSA who take Mathematics as a subject:

i. *Main strata.* The four education departments: Department of Education and Culture, Administration: Parliament; Department of Education and Culture, Administration: House of Delegates; Department of Education and Culture, Administration: House of Representatives; and the Department of Education and Training were taken as the main strata. Pupils from each education department were drawn proportionately so that the number of pupils per main stratum was proportional to the total number of pupils in each department, with a minimum of 500 pupils per department.

ii. *Strata.* When compiling the random sample the main strata were divided proportionally into two strata for urban and rural schools. The medium of instruction was used as a dividing factor in each of the strata. (Only single medium schools were included in the investigation.) An attempt was made to include pupils receiving instruction through the medium of Afrikaans and English respectively in the ratio according to which Afrikaans and English speaking pupils occur in the population.

Information on the theoretical sample is given in the table below.

	RURAL			
	Afrikaans		English	
	Boys	Girls	Boys	Girls
DEC, Admin:Parliament				
OFS	16	16	--	--
Tvl	170	160	53	50
Cape	31	30	15	21
Natal	--	--	29	32
DEC, Admin:H. o. D.				
Tvl	--	--	23	11
Natal	--	--	156	78
DEC, Admin:H. o. R.				
Cape	303	152	--	--

		URBAN			
		Afrikaans		English	
		Boys	Girls	Boys	Girls
DEC, Admin:	Parliament				
	OFS	8	8	32	32
	Tvl	235	235	283	282
	Cape	31	30	200	186
	Natal	22	20	140	130
DEC, Admin:	H. o. D.				
	Tvl	--	--	68	34
	Cape	--	--	52	26
	Natal	--	--	516	258
DEC, Admin:	H. o. R.				
	Tvl	82	41	--	--
	Cape	67	33	--	--
	Natal	--	--	100	50
DET					
	PWV	--	--	750	530
	N Tvl	--	--	40	30
	Highveld	--	--	25	20
	Natal	--	--	45	40

The schools in each main stratum were arranged according to the Serpentine method with province, demographic location and medium of instruction as factors as well as the number of Standard Nine pupils taking Mathematics as a subject. Class lists with an indication of sex and the grade at which Mathematics was being taken were obtained from the schools. The number of pupils included by every school was divided firstly in relation to sex and secondly on the basis of level. Depending on the number of pupils required, every second, third, fourth and so on pupil within each of these categories was indicated. The test was administered in every school by the mathematics teacher or a person nominated by the principal.

The test results of 4 635 pupils from 106 schools could be used for analysis. From the results a global picture of the nature and extent of mistakes was formed and general tendencies were identified.

3.3.2 The final test and application

The categories and groups used in the pilot study were retained. From the 237 items used in the pilot study 162 were selected for the final test. Each group consisted of a total of four to six items, except in some cases where more items were included for correlations. The reasoning behind this is that if a pupil makes a mistake it may be just a careless one which s/he will immediately be able to identify. If a pupil makes two mistakes in a group however we may suspect that there could be a problem. If three or more mistakes are made in a group, we may assume that there is a lack of the aspect being tested.

The items were again distributed at random through the subtests and were incorporated as follows into the final test battery:

The test battery consisted of two parts, TEST A and TEST B. TEST A comprised three subtests of thirty multiple choice items each. TEST B comprised two subtests of thirty multiple choice items each as well as one subtest of twelve open items. It was decided to include as many multiple choice items as possible because of the practical problems associated with the size of the sample. The aim of the open items was to include questions for which explanations had to be given by the pupils. The division of the test battery into TEST A and TEST B was necessary because the pilot study had shown that the test as a whole would be too long.

The 150 multiple choice items were answered on a computer answer sheet and marked with the aid of an optical mark reader. The same programme used for the item analysis of items in the pilot study was used again. For the analysis of the results which is discussed fully in the next chapter, particular attention was paid to the response frequencies of the distractors. The open items were marked by hand and the results were mainly used for interpretation purposes.

Duration of the test. Diagnostic tests should not have time limits. During the pilot study the researcher could however get an indication of the average time required and the anticipated duration of TEST A and TEST B was indicated in the instructions. Testers were asked to allow pupils to write until at least ninety per cent had completed the test. This criterion applied both to TEST A and TEST B.

A break of twenty minutes was allowed on completion of TEST A. The TEST A booklets were taken in before the break.

The test venue. The venue had to be free from disturbances and any mathematical material such as pictures and illustrations had to be removed from the walls.

Pocket calculators. No pocket calculators were allowed for any part of the test.

The test material. Each pupil had to have only a soft pencil. Each pupil received a computer answer sheet together with the test booklets. Subtests one to five had to be answered on the computer answer sheet while the open items of subtest six were answered in the test booklet itself.

Comments on the test. The testers were requested to write down any comments together with the time allowed for TEST A and for TEST B.

3.3.3 Additional information

The Mathematics mark which each test candidate received in the formal examination at the end of Standard Eight had to be entered on the class lists.

3.4 VALIDITY

The validity of a measuring instrument (test) may be defined as the extent to which it succeeds in measuring what it claims to measure. Ary, *et al.* (De Wet, Monteith, Venter and Steyn, 1981:145) caution that validity should not be regarded as a general characteristic of a measuring instrument. Validity relates to a specific task and the evaluation of the validity of a measuring instrument should therefore always be done with reference to a specific use of it. Validity therefore actually means that a test must do what it sets out to do. A test might be highly valid in respect of one aim and highly invalid in respect of another. Instead of referring to the validity of a test, one should rather talk about the validity of a test for a specific application (see Huysamen, 1986:35).

A distinction is usually made between various types of validity. The most usual division is that of the American Psychological Association (see for instance Van den Berg and Vorster, 1982:156):

criterion–related empirical validity,

construct validity and

content validity.

Each of the different sorts of validity has specific relevance for a specific aim. Since the aim of this study was to determine to what extent pupils satisfied certain teaching objectives, content validity was the most important. The content validity of the test used in this study was furthered by a systematic analysis of the universe of tasks so that the important aspects were covered. The researcher compiled the items with a view to educational and content aspects after a thorough study of school syllabuses and subject contents plus an intensive study of the literature. As has been mentioned, the items were critically discussed and evaluated on compilation by an internal committee consisting of persons with the necessary competence and experience.

3.5 RELIABILITY

The reliability of an instrument of measurement refers to the consistency with which it measures or as Mulder (1981:210/211) puts it " ... the validity of a candidate's scores in the same test on different occasions ... "

People's behaviour differs across times and situations and so it will be found that if the same people are subjected to the same test on more than one occasion they will not generally make the same responses to every item on every occasion. The reliability of a test gives an indication of the extent to which individual differences in test results are attributable to real differences in the attribute being tested or can be put down to fortuitous errors. The reliability of a test is in close relation to the measurement error which gives an indication of the extent to which a pupil's marks fluctuate from one application to another — the greater this error the lower is the reliability of the test. The score which a pupil receives for a test consists, according to the classical testing theory model, of two components; a real score and an error score.

Perceived score P = real score R + error score E.

A candidate's score in a test is the perceived score. The smaller the error score, the closer the perceived score will be to the real score and the higher the reliability of the test.

De Wet, Monteith, Venter and Steyn (1981:136/137) mention the following factors which can influence the reliability of a measuring instrument:

a. **The length of the measuring instrument**. The more items included in the test, the higher its reliability.

b. **The heterogeneity of the testees**. The more heterogeneous the sample of candidates, the higher the reliability of the measuring instrument.

c. **The abilities of the testees**. The testees' abilities influence the reliability of the measuring instrument. If a test is too difficult the testees will guess at the answers, and if on the other hand it is too easy many of the testees will answer all the items correctly. Both of these possibilities will result in lower reliability. A highly reliable test will therefore be able to discriminate between strong and weak testees.

d. **The technique used to determine the reliability of a measuring instrument**. Various methods and techniques can be used to determine the reliability of a measuring instrument and consequently various reliability coefficients are arrived at.

Since the test was administered once only, its reliability cannot be determined by means of the test–retest, parallel forms or dichotomous method. Kuder and Richardson have however been developing a number of formulas since 1937 for estimating the reliability of a test without making use of correlation methods. These formulas are in very general use and save time because they require only one application of a test. Van den Berg (Van den Berg and Vorster, 1982:126) summarizes the limitations of the Kuder–Richardson formulas (K–R formulas) as follows:

i. The measuring instrument must be a test consisting of dichotomous items.

ii. The test must be a test of strength, that is, the emphasis must fall on the testees' ability to answer a question correctly rather than on the speed with which the question is answered.

iii. The test scores and the item scores must meet the demands of the classical test model.

In this study the K–R 8 and the K–R 14 formulas were used to estimate the reliability coefficient of the test. They are as follows:

K-R 14

$$\rho_{xx} = \left(\frac{{\sigma_x}^2 - \sum p_i q_i}{(\sum \sqrt{p_i q_i})^2 - \sum p_i q_i}\right)\left(\frac{\sum \sqrt{p_i q_i})^2}{{\sigma_x}^2}\right)$$

K-R 8

$$\rho_{xx} = \frac{{\sigma_x}^2 - \sum p_i q_i}{2{\sigma_x}^2} + \sqrt{\frac{\sum {\rho_{it}}^2 p_i q_i}{{\sigma_x}^2} + \left(\frac{{\sigma_x}^2 - \sum p_i q_i}{2{\sigma_x}^2}\right)^2}$$

where

ρ_{xx} = reliability coefficient of the test,

${\sigma_x}^2$ = variance of the test,

ρ_{it} = correlation between item i and the test score,

p_i = difficulty value of item i, and

$q_i = 1 - p_i$.

3.6 OBJECTIVITY

Subjective prejudices were eliminated by having most of the test marked by the optical mark reader. The open items were marked by hand but had been formulated in such a way that the answers had to be either right or wrong. The results were processed by means of a computer.

3.7 FACTORS WHICH MIGHT INFLUENCE THE RESULTS OF THE INVESTIGATION

No test results can be considered in isolation from the testees' emotional condition, intelligence, circumstances and so on. The test moreover mainly used items with five options, so that all the possible answers given by all the pupils could not be accommodated. When interpreting the results it must also be taken into account that pupils from different population groups wrote the same test.

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CHAPTER 4
RESULTS

4.1 INTRODUCTION

The results are given in two sections, a qualitative and a quantitative analysis of errors and the relation between the mastering of basic concepts and achievement in Mathematics.

An increasing interest in the classification of errors is found in the literature over the past few years. Error analysis has appealed to the imagination and although it is perfectly understood that no error is exclusively attributable to one cause — errors in Mathematics are the result of complex processes and are interwoven with the pupil's behaviour as a totality — models for the classification of errors are increasingly to be found.

4.2 QUALITATIVE AND QUANTITATIVE ANALYSIS OF THE RESULTS

In this section we discuss the results of the empirical investigation as described in the previous chapter. In the analysis of the results, the main focus was on the number of pupils who made a particular error. Errors made consistently by individuals, such as $3 \div 10 = 1.3$ and $7 \div 10 = 1.7$, are regarded as unique individual errors caused by misunderstanding. These pupils need individual attention. The aim of the study is not primarily to collect misunderstandings on the part of individuals but to try and form a global image of pupils' general knowledge and skills in the handling of fundamental mathematical concepts and principles. The results are in most cases also supplemented with explanations of the possible causes of identified errors. An attempt was made to integrate facts rather than simply to provide a collection of them.

The Humans Sciences Research Council does not permit the publication of test items. The researcher was however granted permission to make use of equivalent items to serve as examples of types of questions.

4.2.1 Fundamental concepts, principles and skills

Sixteen topics were included in this category in an attempt to get an indication of pupils' knowledge of essential aspects of Mathematics. The sixteen topics, which were not intended to be comprehensive, are given in the following table together with the average difficulty and discrimination values obtained. To facilitate the analysis the topics have been arranged in order of difficulty.

	Average Difficulty Value	*Average Discrimination Value*
Priority of operations	21,2	0,317
Graphs	21,5	0,241
Inequalities	31,4	0,248
Substitution	37,6	0,366
Roots	38,5	0,358
Factorization	39,0	0,290
Formulas	40,6	0,316
Decimal numbers	41,4	0,336
Impossible problems	41,5	0,294
Principal operations	42,0	0,269
Exponents	43,0	0,368
Fractions	44,5	0,400
Solving equations	48,1	0,336
Percentages	51,4	0,365
Ratios	56,6	0,318
Unnecessary information	57,7	0,204

Each of the respective topics will be discussed in the order in which they appear above. Three aspects are focused on in analysing the results: the difficulty value, the discrimination value and the percentage of pupils choosing the most popular distractor. If the value of the third category is higher than 50 and the answer to the distractor is uniquely obtained, we speak of a *typical error*. A discrimination value of 0,250 and higher (see Van den Berg and Vorster, 1982:86) is usually regarded as suitable for achievement tests. Although the test for this study was chiefly diagnostic, it may in a certain sense also be regarded as an achievement test — just as any achievement test also has diagnostic value. In any event the discrimination values obtained may be regarded as relatively high, in other words the items discriminate well between pupils who generally did well in the test as a whole and pupils who generally did badly. In some instances reference is also made to the other distractors but the above usually gives a good indication of tendencies.

a. **Priority of operations**

By priority of operations is meant the order in which operations are carried out if no explicit indications such as brackets are given. The analysis will be made firstly on the basis of equivalent items. Item A takes the form $12 + 18 \div 6 - 4$. Item B is the identical equivalent of this with letters replacing numbers. Item C takes the form $\frac{1}{3} + \frac{1}{6} \times 7$ and Item D is the letter equivalent of item C. Item E is equivalent to the last group as far as the number of calculations is concerned, but without fractions and with a combination of letters and numbers: $x + x \times 6$.

The following table summarises some statistics concerning the items:

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	18,4	0,362	73,2
B	7,6	0,419	61,9
C	22,2	0,428	40,1
D	23,2	0,317	31,7
E	12,0	0,419	58,3

Three of the items indicate typical errors. Pupils fared considerably worse in Item B than in Item A. The most popular choice for Item A was simply to do the calculations from left to right to arrive at the answer 1. Because 73,2 % of pupils made this error, it may be regarded as a typical error with a particularly high frequency. The pupils who did well in the test as a whole chose the correct answer. The author believes that this typical error is caused mainly by two factors. Firstly one suspects that pupils have not practised this kind of calculation without brackets. Pupils are probably drilled to *always* "remove the brackets" (i. e. perform those operations which are enclosed in brackets) first, but the teachers do not sufficiently show them why it is necessary. Consequently the pupils *always* remove their brackets like automata without thinking about the underlying reason for doing so. The second factor is closely related to the first. In primary school pupils do calculations in isolation from each other and when the calculations are combined, brackets are used. The farreaching implications of this are obvious.

As we have said, Item B was the letter equivalent of Item A. The options were recorded in exactly the same sequence, and instead of numerical answers the operations with letters which would lead to the corresponding numerical answers were presented. The choice which corresponded with the typical error in Item A was also as expected the most popular choice for Item B. A total of 61,9 % of pupils selected this answer as the correct one. An interesting observation is that only those pupils who did well in the test as a whole answered this item correctly. The pupils who got Item A right had an average achievement of 17,9 % above the test average; for those who got Item B right it was 33,8 % above. For Item A the frequency distribution for the other distractors was not noticeably different, while the second most popular choice (25,2 %) for Item B was $x+y \div z-k = \frac{x+y-k}{z}$. This distractor seems to have been chosen on the assumption that the first two terms are added together and divided by z. The resulting fraction must then be reduced by k which is then entered in the numerator without further ado.

Item C and Item D are equivalents of each other in the same way as Item A and Item B. Unlike the previous case there was not any difference worth mentioning in the difficulty values of the two items. As far as sequence of working was concerned, exactly the same tendency showed up here as in the previous case. A total of 40,1 % and 31,7 % of pupils for Item C and Item D respectively worked from left to right without taking into account the order of operational priority, while the pupils who did well in general selected the correct answer in both cases. Item E was as we have said to a considerable extent equivalent to Items C and D. The 58,3 % of pupils who chose a specific distractor indicates a typical error. It was the identical one mentioned before: $x + x \times 6 = 12x$ instead of $7x$. As before the pupils who generally did well in the test once again chose the correct answer.

Item F took the form of $5(x + y)^2$. On the analogy of the above one would expect a typical error of multiplying by 5 before squaring. The statistics for this item, with a difficulty value of $43,8$ and a discrimination value of $0,359$, do not however reveal this. The most popular distractor was $5x^2 + 5y^2$, which 27,6 % of pupils regarded as the correct answer. This makes it clear that the pupils realized that they had to remove the brackets first, but that regression then took place and led to them forgetting the middle term. Once again the high achievers selected the correct answer.

The final item in this section took the form *If there is a 10 % cash discount and 15 % tax, what will the most advantageous choice be for a cash purchaser?* The choices were:

A. First 10 % discount and then 15 % tax.

B. First 15 % tax and then 10 % discount.

C. Either A or B.

D. 5 % tax only.

E. Any of A, B or D.

The difficulty value of the item was 14,9 and the discrimination value $-0,038$. This implies firstly that many pupils (85,1 %) answered the item incorrectly and secondly that the pupils who generally did well in the test answered the item incorrectly. The most favoured distractor was D (25,1 %) — which was also mostly chosen by those pupils who generally fared well — followed by A (20,9 %), E (19,7 %) and B (19,4 %). The distribution of choices among the large percentage of wrong answers would seem to indicate that many pupils based their guesses on face value. The most attractive–looking distractor was chosen without the pupils thinking properly or realizing that both workings involve the calculation of percentages and therefore are on the same priority level. The correct answer could have been reached via the following reasoning:

Ten percent discount means that $0,9$ of the price is paid.

Fifteen percent tax means that $1,15$ of the price is paid.

Choice A: $0,9 \times 1,15\times$ price.

Choice B: $1,15 \times 0,9\times$ price.

The order of working is thus unimportant and the reason is that both the workings are on the same priority level.

Choice D: $1,05\times$ price.

For choices A and B the product of $0,9$ and $1,15 = 1,035$ which is less than the $1,05$ of choice D.

Even if the pupils selected an amount, say R100, as the price they could have done simple calculations.

The results show that pupils do not know the sequence in which calculations must be done. If more than one operation has to be performed and no brackets have been used, pupils simply work from left to right. A possible cause of this is that the operations are probably done in isolation from each other. Pupils first learn how to add and multiplication comes considerably later. When both operations appear in the same problem, brackets are used for grouping. Pupils are probably instructed without any explanation ALWAYS to remove the brackets or they have forgotten why workings are done in a specific order or they did not understand it well. This *rule* is followed slavishly and the pupils get the correct answers to the problems to be solved. The teachers assume that their pupils have grasped the concept because they make no mistakes in their calculations. Knowledge of the priority of operations is taken for granted.

b. **Graphs**

If one considers how important graphs are in the senior secondary phase in particular, the difficulty value of 21,5 is almost frightening. To make matters worse, only a straight line graph was used. The four items took the following form:

Item A: The equation of the straight line through the origin and the point $(3 : 9)$ is ...

Item B: The equation of a straight line had to be found from a graphic representation.

Item C: The equations of two straight lines in two unknowns (x and y) were given. Pupils had to say how many intersections the lines had — different coefficients were given but such that the gradients were the same and the lines therefore parallel.

Item D: $y = \frac{x}{3}$ is the equation of a ...

The following table summarizes a few of the statistics for these items:

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	21,5	0,355	55,2
B	22,4	0,286	33,9
C	22,6	0,209	32,7
D	18,0	0,114	36,5

The statistics show that a typical error was made in Item A. The most popular choice (55,2 % of pupils) was $y = 3x + 9$. Pupils therefore simply substituted the values of the point $(3; 9)$ in the equation $y = mx + c$ without considering the origin. The mistake may have arisen from pupils deciding that the point $(0; 0)$ had no influence on the equation because of the noughts. They therefore only considered one point and the 9 was assumed to be the y-intercept. Carelessness may also have been a factor but seeing that pupils who on the whole did well in the test chose the correct answer, this may be regarded as a secondary cause.

In Item B the equation of a straight line had to be obtained with one point and the angle (45°) made by the line and the Y-axis. Pupils' inability to do this is reflected in the difficulty value of 22,4. Pupils had especial difficulty in expressing the gradient as a number while the y-intercept gave them fewer problems. The good achievers chose the correct answer.

There are two possible causes for the wrong answers in Item C. The 24,8 % of pupils who said that the two lines had one point of intersection probably knew that two different straight lines can have at most one common point but thought that they always had to have one point of intersection, or simply omitted writing the equations in standard form. The 32,7 % who said that there were two points of intersection may have based their choice on the fact that there were two unknowns in the equations.

Although the discrimination value of Item D was lower than those of the other items, the pupils who generally fared well in the test did choose the correct answer. The most popular distractor (36,5 %) gave the answer as *hyperbola*. It is obvious that these pupils were confusing the equation $y = \frac{x}{4}$ with the equation $y = \frac{4}{x}$. There may be several reasons for this — carelessness, pupils not knowing that $y = \frac{x}{4}$ is the same as $y = \frac{1}{4}x$, hyperbolas had been more recently learnt at school, and so on.

Pupils had fewer difficulties in sketching graphs if the equations were given than in finding the equations from given information. The reason for this lies in the fact that pupils follow certain steps when sketching graphs. If a straight line is to be drawn, the equation is written in standard form and the gradient-intercept method (for instance) is used. If a parabola is to be drawn, the equation is written in standard form ($y = ax^2 + bx + c$), the roots calculated by letting y equal nought and then factorizing or using the quadratic formula and so on. Pupils therefore follow certain set steps without really thinking about what they are doing. If the problem is inverted however so that the equation has to be found, greater demands

are placed on insight and understanding because problems can be set in a wide variety of ways. For a straight line graph for example a sketch could be given, or two points, or one point and the Y–intercept, or one point and the gradient.

c. **Inequalities**

The items in this group required pupils to find the solution sets of an inequality such as $4x - 15 > 16$, to describe a shaded area and to manipulate inequalities like $2 - x < 11$ and $4x - 2y > 7$. Generally speaking pupils had difficulty with all the items (average difficulty value of 31,4) but the top achievers consistently chose the correct answers (see also the average discrimination value of 0,248). No typical errors as defined for this study were found and the distractors generally speaking had comparable frequencies. The third item for instance gave the following statistics: distractor A – 17,4 % , distractor B – 20,1 % , distractor C – 20,5 % , correct answer D – 23,2 % and distractor E – 18,8 % .

Pupils treat $<$ and $>$ like $=$. This procedure leads to incorrect answers. In a problem like: "Solve for x if $\frac{8}{x-3} \leq 2x$" pupils multiply both sides of the $\leq$ sign by the denominator $(x - 3)$ of the fraction and after factorizing give the answers $x \geq 4$ and $x \leq -1$ instead of the correct answers $x \geq 4$ and $-1 \leq x < 3$. In most cases pupils stick to their wrong answers without checking them. If they had substituted -2 for instance in the original inequality they could have found out that their answer was wrong.

d. **Substitutions**

The replacement of unknowns by certain values is one of the most basic skills in Mathematics. This skill is often closely linked to the manipulation of formulas in which transformations are usually required. The trends appearing from the items will be discussed briefly with the aid of the table below.

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	38,2	0,382	23,2
B	36,3	0,485	22,9
C	47,3	0,477	17,5
D	28,5	0,416	46,3
E	12,3	0,069	45,3

Item A required substituting in the equation of the straight line $y = ax + b$ the value of the Y-intercept and one other point that the line passes through. Item B had the form *If* $x = \frac{1}{3}$ *and* $x = 5 - \frac{y}{3}$, *solve for* y. In both cases the top achievers chose the correct answer, and the other distractors — with one exception — showed varied frequencies. In Item A the most popular distractor was arrived at by substituting the Y-intercept's value in y and the point's coordinates in x and b. The error, which is not a typical one, seems to have been made by associating the values of the Y-intercept directly with the y symbol in the equation. Since the value of a had to be calculated, the coordinates of the given point could only have been put in the places of x and b. In the case of Item B the most popular distractor was obtained as follows: $x = 5 - \frac{y}{3}$, therefore $3x = 5 - y$. Since $x = \frac{1}{3}$ it follows that $3(\frac{1}{3}) = 5 - y$ so that $y = 4$. The error may thus be attributed largely to regression when the pupils were confronted by too much information at the same time.

Item C took the form *What is the value of* p *if* $-\frac{3}{4}p = 1$? The frequencies of the distractors were relatively similar (in the vicinity of 15 % each) and small in comparison to the difficulty value. The pupils who did well in the test as a whole chose the correct answer.

In item D there was an error which almost met the requirements of a typical error. The item was formulated as follows: *What is the value of* x *if* $x = a + b$ *and* $4x - a - b = 48$? The most popular distractor (46,3 %) gave the answer as 12. The pupils therefore divided 48 by 4 and in the process forgot about a and b. As in a previous case we may suspect that the cause can be put down to regression. Pupils who on average scored 15,8 % above the average of all the pupils in the test answered the item correctly. Item E was of the same type as Item D except that fractions were included in the given equations. Exactly the same type of error had the highest frequency and the lower difficulty value may be the result of the fractions that were included.

e. Roots

The items on this topic included only square roots. As before our discussion of equivalent items is preceded by statistics.

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	45,4	0,399	18,3
B	41,7	0,430	15,4
C	11,5	0,080	51,6
D	33,8	0,375	29,9
E	23,4	0,316	36,3
F	65,4	0,476	13,1
G	47,6	0,483	30,8
H	11,9	0,302	68,7

Item A took the form of $\sqrt{3}-\frac{5}{\sqrt{5}}=\ldots$ and Item B was its letter equivalent. In both cases the frequencies of the distractors centred around 15 and the good achievers selected the correct answer. The difficulty value of Item B was slightly lower than that of Item A which may be attributable to the use of letters.

In item C the pupils had to perform the following kind of calculation: $\sqrt{12}-\sqrt{3}=\ldots$ The choice of 3 as the correct answer was made by 23,2 % of the pupils. The largest number of pupils, 51,6 % , did not select any of the numerical answers (2, $\sqrt{3}$, 3 and 1) and indicated the fifth choice — none of the above — as correct. These pupils may have realized that the square roots of 12 and 3 are not rational numbers but they did not indicate 3 as the correct answer. It may therefore be accepted with reasonable certainty that pupils realized that the 3 cannot simply be subtracted from the 12 but did not notice that 12 can be rewritten as 4×3 and simplification applied to arrive at the correct answer. (Bear in mind that pocket calculators were not allowed.) Since the distractor with the highest frequency could have been chosen on the basis of a number of arguments and not one unique error, this is not regarded as a typical error. In Item D ($\sqrt{2x}+\sqrt{2x}=$) 29,9 % of pupils did give the answer as $\sqrt{4x}$ but the frequency is too low for a typical error. The highest frequency of pupils did choose the correct answer. Similarly the majority of pupils (per choice) selected the correct answer for Item E, $3+\sqrt{x}=2\sqrt{x}$ (solving the equation) while the rest of the choices had varied frequencies.

The high difficulty value of Item F where the square roots of some whole numbers had to be calculated indicates that pupils were capable of doing this. In Item G the whole number was replaced by a mixed fraction and the lower difficulty value suggests that this caused problems for the pupils. The most popular distractor for the item ($\sqrt{4\frac{1}{6}}$) was $2\frac{1}{3}$. The pupils did not transform the fraction and calculate

the square roots of the denominator and the numerator, instead halving the four and the six — the square root of 4 could also have been worked out. Apart from the 47,6 % of pupils who answered the item correctly and the 30,8 % who chose the distractor mentioned, the frequencies of the other distractors were very low.

The final item to be discussed in this section involved the simplification of a trinomial within a square root. Each of the terms of the trinomial were squares and the trinomial could also be written in the form $(ax + b)^2$. The correct solution which was of the form $ax + b$ was chosen by only 11,0 % of the pupils and their average achievement was 19,5 % above the general average for the test as a whole. This means that the bright pupils indicated the correct choice. The most popular choice may be regarded as a typical error since 68,7 % of the pupils selected it; it was to calculate the square root of each term separately. Many pupils in other words think that $\sqrt{z^2 + y^2 + z^2} = x + y + z$. This may be ascribed to the fact that pupils confuse $\sqrt{x + y}$ with $\sqrt{x \times y}$ (interference) in which the square root can be "divided". Another possible cause is that pupils formulate their own rules on the basis of what they see — it is easier to make the mistake than to factorize the trinomial. It seems that pupils can solve problems containing square roots if the problem is simple or analogous to an example discussed in the classroom. As soon as an item demands more than one thought process large numbers of the pupils fail to answer it correctly. Regression is a possible reason why pupils choose the easy way out.

f. Factorization

The factorization of trinomials and the extraction of common factors may be regarded as of cardinal importance for the practice of Mathematics. Sketching parabolas for instance and solving equations are just two examples of the frequent need for factorization. The following table gives some of the statistics necessary for a discussion of the item results.

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	36,0	0,203	22,5
B	45,1	0,276	18,2
C	59,3	0,308	16,4
D	23,4	0,254	24,9
E	18,6	0,409	58,2

A difficulty value of 86,0 was found for an item of the form $x^2 + ax + b$ (the statistics of which are not given here). This means that pupils do not in general have difficulty in factorizing the *standard* trinomial. This difficulty value reduced significantly with the use of negative coefficients (Item A). The change to the form $k^{2p} - 2k^p - 15$ (Item B) also led to a markedly lower difficulty value. In both cases the pupils who generally did well chose the correct answers and the frequencies of the distractors were of the same order. The obvious conclusion is that the use of negative numbers and non–standard forms (such as k^{2p} instead of x^2) made the items more difficult. Pupils apparently do not see that the form of the trinomial is unchanged.

The difficulty value of 59,3 for item C (extracting a common factor from two terms) shows a fairly general mastery of the technique. In Item D the difficulty value is considerably lower than in Item C. In this item the common factor which had to be extracted from the binomial was a fraction. Once again the good achievers chose the correct answer.

A typical error was made in Item E. This item comprised the simplification of $\frac{6x+9}{2x+3}$ — the common factor 3 in the numerator had to be extracted. The distractor which 58,2 % of the pupils chose was $3x + 3$. Pupils are thus clearly showing a lack of understanding that the denominator has to be treated as a unit, and cancelling blindly. The average achievement of the pupils who did answer the item correctly was 20,2 % above the average for all the pupils tested. The high discrimination value also indicates that the good achievers did not make the typical error. Vorster (1970:96) found an average difficulty value of 11 for a similar item while some 49 % of the pupils chose the corresponding distractor. He points out that the solution is simple if the numerator is factorized first but that the pupils forget this because it does not get drilled into them. The carelessness which leads to poor achievement is ascribed to autonomous processes and interference.

Observation: *The typical error was made because the fraction was incorrectly handled but the item was included under factorization because the principle of extracting a common factor which leads to the correct solution belongs under factorization.*

g. Formulas

Formulas are discussed under two headings, knowledge of formulas and the manipulation (transformation) of formulas. The table below summarizes the average difficulty and discrimination values of the items in the two groups:

	Average difficulty value	*Average discrimination value*
Knowledge	39,6	0,350
Manipulation	26,4	0,303

In the first knowledge item pupils had to give the correct form of the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This item, which involved no more than mere recall of ready knowledge, was answered correctly by 57,9 % of pupils. No pattern could be found in the frequencies of the distractors. Three items involved the calculation of the perimeter or area of familiar figures (circle, rectangle and triangle) from drawings which gave the necessary information. The item in which the length and breadth of a rectangle was given and the *perimeter* had to be calculated had a difficulty value of 48,2. The only distractor which had a high frequency (42,8) was the one in which the *area* of the rectangle was given. This error may be attributable to the fact that pupils are more used to calculating areas than perimeters and consequently simply made a careless mistake. It is also possible that they may have read the question too quickly. Pupils read into a question what they want to see. The same tendency was found in an item for which a sketch of a circle was given with the radius indicated on it and pupils were instructed to calculate the circumference of the circle. A total of 36,0 % of pupils calculated the area. In the last item to be discussed under knowledge of formulas a right–angled triangle was given with the hypotenuse and one other side. Pupils had to calculate the area of the triangle. Only 27,7 % of the pupils managed to give the correct answer. The distractor which most of them chose (23,1 %) was the sum of the two given numbers and the next most favoured distractor (20,3 %) involved the calculation of the third side and the sum of all three sides (perimeter). It would appear therefore that pupils (43,4 %) wanted to calculate the perimeter. This may be partly attributable to regression since the remaining side had to be calculated before the other calculations could be done.

The lower average difficulty value of the items in which formulas had to be manipulated indicates that the principles according to which this is done have not been properly mastered. Each option attracted about 20 % of the pupils and no typical errors surfaced. It would appear that many pupils often guessed on the basis of face value. An item which included an exponent had the lowest difficulty value at 12,9. It was however clear from the choices that the pupils were confusing the

two principles $a = b - c \Rightarrow a + c = b$ and $a = \frac{b}{c} \Rightarrow ac = b$. De Wet (1966) and other authors found the same phenomenon in their empirical studies.

Butler and Wren (De Wet, 1966:156) attribute pupils' problems with the manipulation of formulas to the fact that pupils are not always deliberately led to notice the basic principles and concepts which lead to the solutions.

h. **Decimal numbers**

Statistics for the items on decimal numbers are given in the table below.

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	50,3	0,314	45,2
B	46,6	0,405	46,1
C	40,1	0,314	23,4
D	27,2	0,311	39,5

In all four items the pupils who generally did well in the test chose the correct answers. Item A took the form $0,4 \times 0,2 =$... Of the 49,7 % of pupils who answered the item wrongly, 45,2 % chose the distractor 0,8. This error can almost be regarded as typical and may be attributed to the fact that pupils do this sort of calculation with the aid of pocket calculators. (Pocket calculators were not allowed in the test.) Pupils may have mastered the principle in the primary school phase but in the secondary school phase they lose to some extent the ability to do simple calculations in their heads. The same phenomenon appeared in Item B. This item, $12,16 \div 4$, had 46,1 % of pupils choosing its most popular distractor, 3,4. In Item C a percentage had to be written as a decimal. Pupils who answered the item wrongly had obvious problems about places. The statistics for Item D show that the same problem as in Item C made its appearance again. Pupils had to simplify $\sqrt{0,0049}$. The distractor chosen most often was 0,007 – the square root of 49 was calculated and the place values were ignored.

To summarize, the pupils in general revealed an inability to do calculations with decimals in their heads. The calculations using figures appear not to have been a problem, but the place values were. The cause of this may be the use of pocket calculators — pupils do even the simplest calculations with their pocket calculators instead of using them mainly for longwinded, difficult calculations.

i. **Impossible problems**

The aim of including problems for which solutions could not be obtained was to try and determine whether pupils could see when too few facts were available or other information made the finding of a solution impossible. In general it was found that pupils were capable of identifying problems in which essential information was lacking and also of saying what information was missing. Since the items included were divergent, only two examples (with the highest and lowest difficulty values) will be given. First example: ***A boy buys 10 books. The books with the hard covers cost five rand each and the books with the soft covers cost three rand each. How much did the books cost the boy in total?*** This item had the highest difficulty value at 76,9. The frequencies of the distractors centred around 6 which does not point to any specific problem. On the basis of this and other similar problems we may conclude that pupils did not in general have serious problems with this kind of item. The item with the lowest difficulty value was formulated as follows: ***A board is sawn in two. If one part is three fifths of the total length of the plank and the other part is 15 cm longer than the first part, how long is the plank?*** The difficulty value of 35,8 indicates that this problem was generally found to be more difficult. The frequencies of the distractors were once more of comparable order. No pattern could be found in the answers while the bright pupils chose the correct answers.

j. **Mental calculations**

It can be accepted from the test results in general that pupils do not have difficulty with working in their heads unless there are other factors present to derail them. In general they were found to know their tables and to perform correctly simple calculations with integers. The discussion of items in this section will be based on two of the items in which other factors played a part; their statistics are given below.

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	37,7	0,399	19,4
B	26,8	0,183	41,9

In Item A pupils had to indicate which option was equal to a fraction, say $\frac{a}{b}$. The choices took the form $\frac{a+x}{b+x}$; $\frac{a-x}{b-x}$; $\frac{a \times x}{b \times x}$ and $\frac{a \div x}{b \div x}$. There was no difference worth mentioning between the numbers of pupils who chose the distractors. The low

difficulty value may be attributable to the fraction form for which understanding of manipulations has not been formed properly.

Item B illustrates a basic principle and almost provided a typical error. The pupils had to indicate what has to be subtracted from $a^2 - b^2$ to give the answer nought. Although the pupils who did well in the test as a whole answered the item correctly, 41,9 % of pupils chose the distractor $-a^2 + b^2$. These pupils therefore gave the answer to the addition process. The error they made may be attributable to the fact that pupils merely gave the additive inverses of the given terms and its opposite (as far as the signs are concerned). This phenomenon of ignoring the operation in question was also found in other items — see the discussion on exponents.

In one of the open items the pupils had to write down the largest possible value for x if $x = -3 \pm 4$. Only 38 % of the pupils managed to give the correct answer. The minus sign in front of the 3 caused particular problems. The relevance of such knowledge will be especially obvious if one considers the use of the quadratic formula.

Venter (1978:64/65) believes that the meaningful mastery of the four main calculations is based upon understanding of the distributive rule. Pupils who experience problems with this rule take refuge in recipe methods which have no foundation in insight.

k. **Exponents**

In this section only three of the items included in the test will be discussed. The statistics for these items are as follows:

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	50,7	0,298	38,4
B	8,1	0,295	74,5
C	33,4	0,491	37,5

The fraction $\frac{10^3}{2^3}$ had to be simplified in Item A. The largest number of pupils did choose the correct answer but the distractor 5 was selected by 38,4 % of pupils. The answer to this popular distractor could have been obtained in two ways. The products of the base and the exponent could have been calculated for both the denominator and the numerator and then simplified, or the exponents of

the denominator and the numerator could have been "cancelled". It was found in another item that many pupils followed the former method. In Item B a similar question was included but the exponents were replaced by an x. This item looked like: $\frac{10^x}{2^x}$. In this case the distractor with the answer 5 was chosen by 74,5 % of pupils, which most definitely indicates a typical error. The deduction that can be made from the items in this section is that pupils are inclined to multiply the base and the exponent by each other if they both consisted of figures. If a letter is used in the exponent it is simply cancelled if it is the same letter. This amounts in principle to the same error since pupils make the following simplification: $\frac{10^x}{2^x} = \frac{10x}{2x} = 5$. Further confirmation that pupils are not really in control of the notation of exponents came to the fore in Item C. In this item the pupils had to calculate half of 2^4. Apart from the 33,4 % of pupils who got the item right, the answers of a total of 58,9 % of pupils were divided between two distractors. The distractor 1^2 was chosen by 37,5 % of the pupils and 2^2 by 21,4 % . There is no question here of a typical error as it has been defined, but it is very clear that these pupils do not really understand exponential notation. Routine problems may perhaps be correctly solved in the classroom with the aid of examples but as soon as the pupils are confronted with problems where insight into their acquired knowledge has to be shown, they often fail because of a lack of real understanding.

It was also found that pupils were inclined to divide exponents instead of subtracting them in items like $\frac{x^8}{x^2}$. Vorster (1970:93) ascribes this valency to interference from knowledge of fractions and from cancelling in particular. This phenomenon is especially clear if the numerator's exponent is a multiple of the denominator's exponent.

l. Fractions

The place occupied by fractions in the rank order of difficulty values should not be misunderstood. The objective of items dealing with fractions was to examine fundamental principles and skills. The problems were therefore set very simply. It will appear from the general discussion after the briefer discussions of the separate topics that the inclusion of fractions (and exponents) makes a problem considerably harder. We have previously pointed out that two similar items — one with a fraction and the other without — had drastically different difficulty values. It should also be mentioned that fractions in general provided the best discrimination values (an average discrimination value of 0,400).

In discussing the items on the concept of fractions we distinguish between operations with fractions and simplifications of fractions. The statistics for some items in the former category are:

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	70,3	0,472	20,3
B	62,4	0,466	22,7

Item A is surely the very simplest kind of item that could have been included (unless the denominators were the same). In this item the candidates had to add together two fractions of which the numerators and denominators were single–digit numbers (for instance $\frac{1}{4} + \frac{2}{3} = \ldots$). The only distractor worth mentioning (20,3 %) was the one in which the smallest common multiple of the denominators was not calculated but the numerators and denominators were simply added together. In another item the fractions were replaced by decimals. The difficulty value of this item fell to 24,9. The most popular distractor (45,4 %) was once again the one in which the numerators and denominators were added together. The conclusion to be drawn from this is that pupils have a very strong tendency to add the numerators and then the denominators separately. This tendency increases particularly if the numerators and denominators are not simple numbers. It would appear therefore that pupils have only a vague idea of the concept of fractions and are easily thrown into confusion if any deeper ability to handle them is required. This deduction is supported by findings in other items. One item for which the candidates had to say what had to be added to $\frac{2}{5}$ to give $\frac{3}{7}$ had 37,8 % of pupils giving the answer $\frac{1}{2}$. (The difficulty value for this item was 36,2.)

Item B took the same form as Item A except that the operating symbol was changed. Pupils had to divide one fraction by another. The numerators and denominators of both fractions were single–digit numbers and the $\div$ symbol was replaced between the fractions. The distractor for which the products of the numerators had been calculated was the choice of 22,7 % of the pupils. These pupils knew therefore that products had to be calculated, but forgot to invert one fraction. The item was also correlated with an item in which the numerators were replaced by letters. The difficulty value of this item was slightly higher than that of Item B but the analogous distractor was once more the most popular with 20,6 % of the pupils choosing it.

We may summarize by saying that it would appear that pupils memorize the algorithms but do not understand the principles. For instance, they do not appear to know that the value of a fraction does not change if both the numerator and the denominator are multiplied by the same amount. The ones who do know the principle often confuse it with addition, in other words they reason that if the same number is added to both the numerator and the denominator of a fraction the value of that fraction is not affected. Brueckner (Vorster, 1970:40) found that a lack of understanding was the major source of errors in working with arithmetical fractions. Pupils are easily confused if they cannot recall the algorithm perfectly and then make mistakes by muddling up the principles.

The findings on pupils' ability to simplify fractions are given by way of the following selected statistics:

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	36,6	0,400	38,1
B	38,1	0,364	50,8
C	31,1	0,349	40,7
D	32,1	0,513	36,5
E	43,8	0,577	29,3

The first item (item A) examined pupils' understanding of the denominator and the numerator of a fraction. The item stem and choices from an equivalent item will be given to ensure clarity: $\frac{a-b}{x-y} = \ldots$ Choice 1: $\frac{a}{x-y} - \frac{b}{x-y}$, choice 2: $\frac{a}{x} + \frac{b}{y}$, choice 3: $\frac{a-b}{x} - \frac{a-b}{y}$, choice 4: $\frac{a}{x} - \frac{b}{y}$ and choice 5: $\frac{a-b}{x} + \frac{a-b}{y}$. Choice 4 was chosen by 38,1 % of pupils while the other choices were selected by 8,8 % of candidates or fewer. Choice 4 was therefore more popular than the correct answer which indicates that the pupils in general did not regard the denominator of a fraction as a unit. If the sum total of the incorrect answers is taken (63,4 %) then this tendency indicates a typical error. The author suspected this in advance and checked it by means which included Item B. This item was of the form $\frac{a^2-b^2}{a-b}$. The typical error $a - b$ was the result of cancelling without factorizing. In other words pupils do not regard the denominator as a whole and cancel terms in it with terms in the numerator. Vorster's investigation (1970:94) showed an average difficulty value of 23,2 for this item. A total of some 62,2 % of pupils chose the distractor $a - b$ and Vorster rightly points out their strong tendency to cancel. He confirmed these findings with an item $\frac{a^2+b^2}{a+b}$ for which an average difficulty value of 19,8 was given. In this item

some 62,7 % of pupils regarded the distractor $a + b$ as the correct answer. In Item C as well which required pupils to simplify $\frac{x}{x+y} + \frac{y}{x-y}$, 40,7 % of them selected $\frac{1}{y} + \frac{1}{x}$.

Item D took the form $\frac{a^2+a}{a}$. The fraction therefore had only one term in the denominator and two in the numerator. The distractor with the highest frequency was a^2. The denominator was therefore only divided into the term in the numerator which was the same — but the resulting 1 was left out! Although few items of this nature were incorporated into the test the author is willing to express a strong suspicion that pupils do not really understand what they are doing — consider for instance the low difficulty value of this item which Standard Nine pupils should find easy.

The final item included under fractions was: $\frac{a}{b} + a = \ldots$ Almost half of the pupils gave the correct answer and $\frac{2a}{b}$ was the most popular distractor. What catches the eye however is the exceptionally high discrimination value of 0,577. The average achievement of pupils who did well at the test as a whole was 28,2 % higher than the achievements of the other pupils.

The place occupied by fractions in the list of topics included should be interpreted in the light of the aim of the study. It was very clear from other items that pupils have not really mastered the concept of fractions. If pupils remember the method (recipe) according to which calculations involving fractions are performed, they arrive at the correct answers. As soon as an item tests a concept, however, the same pupils fail. Because various items in this section dealt with simple calculations with fractions, the average difficulty value appears misleading. Many pupils do not know that $\frac{x}{y}$ means that x of the fractions $\frac{1}{y}$ must be added together. In one of the open items for instance pupils were asked which of $\frac{t}{3}$ and $\frac{1}{3}t$ was the largest. Only 35 % of pupils knew that they have the same value while 35 % reasoned that $\frac{t}{3}$ was larger and 30 % favoured $\frac{1}{3}t$. In another open item they were asked whether half of a seventh can be calculated. A total of 28 % of pupils gave the correct answer while 27 % thought the answer is $3\frac{1}{2}$; 24 % said that it can not be done because of 7 being an uneven number, and the balance gave other incorrect answers.

m. Solving equations

Like substitutions the solving of equations must surely be regarded as one of the most fundamental skills in Mathematics. The pupils who cannot solve equations will experience great difficulty in making any progress at all in the subject.

As in the case of fractions the items used in the test were not meant to be difficult. Instead an attempt was made to identify tendencies in the basic aspects. The solving of algebraic, trigonometrical and other types of equations in a way requires exactly the same principles — unknowns are grouped together via manipulations and reduced. Some of these essential skills were investigated by means of items which gave the following statistics:

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	32,9	0,272	39,2
B	76,6	0,202	13,6
C	83,3	0,372	5,2
D	44,0	0,443	30,6
E	40,1	0,497	28,4
F	25,9	0,093	40,2

Item A tested one of the most basic concepts, inadmissible division by an unknown, by means of the stem $y^2 = 4y$. The high frequency (39,2 %) of pupils who gave $y = 4$ as the only solution almost rates this as a typical error. The solution $y = 0$ was thus eliminated.

In Items B and C pupils had to solve equations like $2x - 5 = 17$. The difficulty values indicate that pupils did not in general have difficulty with this. When a fraction is included, as $\frac{2x}{3} = 12 - x$ was in Item D, then the difficulty value drops markedly. In Item E the unknown was put into the denominator of the fraction and an even lower difficulty value was obtained. In Item F the equation was quadratic ($x = \frac{1}{x} - \frac{5}{6}$). Pupils generally found this type of problem to be considerably more difficult. The error of not working out the lowest common multiple of the fractions was made by 40,2 % of them.

The results show that pupils do not have many problems about moving terms correctly across the = sign. The mistakes arise particularly when fractions are included and terms with denominators have to be multiplied to simplify the fractions. Errors are especially frequent when the denominators contain unknowns. Weaker pupils are furthermore inclined also to confuse the two principles $a = b+c \Rightarrow a - b = c$ and $a = \frac{b}{c} \Rightarrow ac = b$ with each other as we have mentioned. Many errors are made when more terms appear or occur in the same problem at the same time.

De Wet (1966:156/157) proposes an alternative method for solving equations since the traditional methods have in his view deteriorated into a series of conjuring tricks. Equations such as $2x = 16$, $\frac{x}{2} = 16$ or $x + 2 = 16$ are solved by using intuitive methods. One looks for the number by which two has to be multiplied, which two must be divided into or to which two must be added in order to arrive at the answer 16. In this way the pupil solves the equation thoughtfully by using inverse reasoning to find the answer. This involves therefore the reverse of the traditional procedure. Instead of using a problem (such as "A certain number is added to two and the answer is 16 so what is the number?") to form equations and then to solve them mechanically, a problem is formed from the equation and solved first intuitively and later by means of inverse reasoning. Using this method means that less use is made of mechanical recipe methods.

n. **Percentages and ratios**

Generally speaking pupils can perform the basic calculations. Pupils found that items like "Calculate 12 % of 50 " and "There are 32 pupils in a class. For every six boys there are two girls. How many boys are in the class?" were easy.

A tendency that did appear was that pupils often do not see a percentage as a ratio (a fraction with the denominator = 100). The stem of an item with the difficulty value of 35,9 was worded as follows: *A farmer has 400 cattle. He sells 30 % of them and increases the rest later by 20 %. How many cattle does the farmer have now?* The most popular distractor was 370, chosen by 31,8 % of pupils, and the second favourite was 390 (20,3 %). A total of 52,1 % of pupils therefore did not work out the percentages but regarded them merely as figures for a number of animals. The same tendency was found to a lesser extent in another item. In this item the pupils had to reduce 20 by 5 %. A tatal of 19,5 % of pupils gave the answer 15 while 21,8 % gave 4 ($20 \div 5$).

To conclude the discussion of items covering these topics, the item with the lowest difficulty value must be mentioned. The stem of the item was formulated as follows: *If 900 square tiles are used to cover a square area of* $1m^2$, *what is the area of one tile?* Only 16,3 % of pupils managed to indicate the correct answer while 48,5 % chose the answer $9cm^2$. Their error, which almost qualifies as a typical error, was simply to reason that there are 100 centimetres in a metre and so the 900 must be divided by 100.

o. **Redundant information**

Redundant information means that more information is given than is needed to solve the problem. The aim was to find out whether pupils could select the essential data from the facts given. To illustrate the types of problems included, we will mention the items with the highest and the lowest difficulty values.

The item which the pupils found the easiest was formulated as follows: *A girl needs 430 Rand. She works for four days at 20 Rand a day and for six days at 30 Rand a day. How much money did she earn?* The difficulty value of 87,9 for this item shows that the pupils were not put off by the amount required. The second item was formulated as follows: *Twelve rabbits eat 48 kg of food in four weeks. If there are 16 rabbits in a hutch and 52 weeks in a year, how many kg of food will four rabbits eat per week?* The difficulty value for this item was 32,7 and the frequencies of the distractors were comparable. Pupils made various errors — some included the 16 rabbits into their calculations, and some the 52 weeks.

The only meaningful conclusion that we can come to is that the more information is used, the more difficulties the pupils generally experience with the problem. Teachers may be teaching their pupils to use all the information when solving problems.

4.2.2 **Mathematization**

Items in this section were divided into four groups: notation, terminology, colloquial language and conversions.

In the following table we give the average difficulty and discrimination values of the items in each group:

	Average difficulty value	*Average discrimination value*
Notation	28,7	0,298
Terminology	35,9	0,153
Colloquial language	46,5	0,406
Conversions	31,4	0,437

We may deduce from the table that pupils find notation to be the most difficult and colloquial language the easiest. It is true of course that language plays a prominent role in conversions and the solving of verbal problems. The values obtained should also be seen in the light of the aim of the study.

a. **Notation**

The findings of the study correspond with the general belief that pupils have difficulty with the symbolism of Mathematics. Even notations which are generally assumed to be familiar will often not be clear to pupils. In one of the open items pupils were asked which was larger, $3x$ or $x + 3$. Only five per cent of the total number of pupils were able to give acceptable answers. A total of 48 % reasoned that $3x$ would always be larger because multiplication *always* gives a larger answer than addition, while 7 % of the pupils reasoned that $x + 3$ was larger because the expression has more terms than $3x$. The most amazing observation was however that 26 % of the pupils were convinced that $x + 3$ and $3x$ are identical in all cases.

The following table gives an indication of the types of symbols used in some of the items as well as the relevant statistics:

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
Exponents	15,1	0,216	36,0
Function notation	22,0	0,322	45,8
Mixed fractions	32,2	0,363	49,9
Set notation	45,4	0,291	18,2

Observation: *The item with the exponent contained a fraction which made the item more difficult.*

Two items which will justify discussion are the second and the third, in which errors appeared which were almost typical. The item on the mixed fraction was "$3\frac{1}{2} =$" and the choices were $3 + \frac{1}{2}$, $3 \times \frac{1}{2}$, $3 \div 2$, $31 \div 2$ and $\frac{6}{2}$. The 49,9 % of pupils who chose the second option as correct represent a typical error. Many pupils think that $3\frac{1}{2} = 3 \times \frac{1}{2}$. It is however necessary to mention that the pupils who generally did well in the test did give the correct answer of $3\frac{1}{2} = 3 + \frac{1}{2}$. We may thus conclude that the item discriminated well between good and weak pupils (see the relatively high discrimination value). The mistake may perhaps be ascribed to confusing ways of writing. Pupils are shown repeatedly that xy means $x \times y$. They then make an erroneous jump to the arithmetical mixed fraction and this gives rise to the typical error.

The second item to be investigated is the one in which function notation is used. The item looked like this: *Given* $f(x) = 2x - 3$, *determine* $f(x + 4)$.Almost half of the pupils identified $2x + 1$ as the correct answer. These pupils therefore interpreted $f(x+4)$ as $f(x)+4$. As in the previous case, the pupils who did well gave the correct answer. The tendency which can be deduced from this information is that the pupils who generally do badly do not really understand the notation. These pupils will quite probably also make mistakes like $\sin(x + y) = \sin x + \sin y$.

In this connection we may also refer to statistics obtained from the formulation of the same item in three different ways:

i. First formulation: Solve for x: $3x - 6 = 36$.

ii. Second formulation: Given $f(x) = 3x - 6$. Find the set $A = \{x/x \in N; f(x) = 36\}$ where N is the set of all whole numbers.

iii. Third formulation: If a number is multiplied by 3 and then reduced by 6, the answer is 36. What is the number?

All three formulations were included in the test with the aim of looking at the frequencies of correct answers. The first formulation of the problem was mastered by 76,2 % of pupils, the second by 43,7 % and the third by 83,3 % . The respective formulations were distributed in the test so that pupils would have great difficulty in spotting that they were actually doing the same problem. The remarkable differences in difficulty values may consequently be largely attributed to the formulations. It is especially important to note that the function notation of modern Mathematics seems to be difficult for the pupils to handle.

b. **Terminology**

The following table illustrates the types of terms used in the items in this section, with their statistics:

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
Roots	33,4	0,311	21,6
Coefficients	43,2	0,310	33,6
Terms	69,1	0,176	17,0
Powers	11,9	−0,140	77,4
Rational numbers	17,6	−0,004	38,0
Squares	15,1	0,326	75,0
Functions	61,2	0,416	16,3
Perfect squares	25,0	0,245	36,2
Quotients	41,0	0,026	42,7
Additive inverses	32,3	0,094	54,7
Distributive law	40,0	0,202	16,8
Reciprocals	51,7	0,222	22,6
Polynomials	64,5	0,140	11,3
Commutative law	35,4	0,043	23,1

It may be seen from the above that there were three items in which typical errors were made. The item *What is the power in* $4x^3$? had the lowest difficulty value of 11,9. This item was therefore wrongly answered by 88,1 % of the pupils! The most popular distractor was 3, chosen by 77,4 % of pupils. The pupils who generally did well in the test also gave this distractor as the correct answer — this lead to a low discrimination value. The reason for the incorrect choice is largely attributable to the language used in the classroom. Teachers talk about "four to the power of three" and this usage leads indirectly to the pupils getting the idea that the 3 is the power. The typical error is therefore almost exclusively attributable to the language used in teaching. (It should be noted, however, that "power" is sometimes used as a synonym for "exponent".)

A second typical error was found in the item *What is the square of* $9x^4$? The difficulty value of 15,1 implies that 84,9 % of pupils gave the wrong answer. Of these 75 % gave the answer $3x^2$. This error may perhaps largely be attributed to the fact that pupils have to calculate square roots most often. It really represents an example of rigid thinking. We cannot ignore the possibility that pupils might have read the question in too much of a hurry. The statistics showed that the pupils who generally did well in the test also chose this distractor. The fact that the calculation was not difficult strengthens this suspicion.

The third typical error to show up was made in the item *What is the additive inverse of* $\frac{2}{5}$? The most popular choice was $\frac{5}{2}$. Of the 67,7 % of pupils who got this item wrong, 54,7 % regarded this answer as the correct one. The statistics indicate that some of the good pupils did choose the correct answer. The error is largely attributable to confusion with the multiplicative inverse. Since the latter is more commonly used for the simplification of expressions and the solving of equations, pupils tend to expect to see it.

The item in which the term *distributive* was used had a difficulty value of 40,0. In this item pupils had to say which of the five choices the distributive law had been used in. In another item the pupils only had to use the distributive law to "remove brackets". There was a difficulty value of 93,4 for this item. This observation was checked with analogous items in which the commutative law was used. (See the above table.) Similar results were found. On the basis of this observation we may therefore say that pupils do not always know the names of the rules they use, but do apply them correctly.

Finally we may refer to the item in which the term *root* was used. This term is of fundamental importance and although it does not fall under typical errors as defined earlier, we should mention that two of the distractors contained the factors of the trinomial. Altogether 32 % of pupils chose these distractors — there is confusion therefore between the terms *root* and *factor*. This finding strengthens the suspicion that pupils often solve problems like the ones in the examples without really knowing what they are doing.

c. Colloquial language

We pointed out in the previous section that the wrong use of language by teachers may be the reason for so many pupils using the word *power* wrongly. In this discussion we will only look at three words. The first word is *square*. In the senior secondary phase in particular this word is used in classrooms almost every day. The results of the item in which this word was used show that pupils generally understand what is meant by it — 19,3 % of pupils multiplied by two. The second word *double* was also interpreted correctly. A typical error was however made in one item. Pupils were asked to double $3x$. Almost all of them knew that they had to multiply by two but 58 % gave the answer $6x^2$. The word was understood correctly, in other words, but there is a tendency to square unknowns. The same tendency was found in another item where pupils had to halve $6x^2$. The term was

understood but the 70,7 % of pupils who answered $3x$ instead of $3x^2$ show that a typical error was being made.

On the basis of the items used we may conclude that pupils generally have a correct understanding of the basic words used in classrooms. The low average difficulty value (46,5) given above should be put down to errors arising from basic principles rather than from language comprehension.

d. **Conversions**

Conversions may be regarded as the fundamental mathematization process because they are concerned with symbolizing ordinary language. Pupils generally had problems with these items. An item like *ten times the product of x and y subtracted from four times the sum of x and y* could for instance be written in symbols by only 29,1 % of pupils. The problems given were divergent and so only two items will be discussed by way of example. These items were selected because of typical errors which came to the fore. The statistics for these two items are given in the table below.

Item	*Difficulty value of the item*	*Discrimination value of the item*	*% of pupils choosing the most popular distractor*
A	26,1	0,158	45,3
B	28,8	0,130	57,5

The two items will first be given and then briefly discussed. The first item, Item A, was formulated as follows: *If p rand are q dollars, how many dollars will a 60 rand jacket cost?* Of the 73,9 % of pupils who got the item wrong, 45,3 % regarded $\frac{60p}{q}$ as the right answer. Item B: *In a class there are four times as many girls as boys. Using G for number of girls and B for number of boys, write down the relation between boys and girls*. This item had a difficulty value of 28,8 and 57,5 % of pupils regarded $B = 4G$ as the correct answer. This typical error shows that pupils seem to think that the B stood for boys and the G for girls. The letters were therefore not interpreted as variables. Clement (Davis, 1984:116) agrees that the error cannot simply be attributed to pupils' tendency to translate word for word; he sees the error as more deep–seated. He and his co–workers confirmed this by changing the word order and the semantic content. Vorster (1970:84 et seq.) observed the same phenomenon. In the item *I am y years old. I am 3 times as old as John. How old is John?* an average difficulty value of 29,8 was found and some 46 % of the

pupils gave $3y$ as the correct answer. Vorster believes that the strong tendency to multiply should be put down to the words "three times". Pupils do not think about the context in which the words are used and react automatically by giving a literal translation. They do not check their conversions.

4.2.3 Dematematization

A few items were included in the test to try and establish whether pupils look at the meaningfulness of their answers.

In one item for instance pupils were asked how many people should be transported in each bus if there were 68 people and three buses available. Some 27 % of pupils gave the answers of the correct type, but only a few saw that there could be many correct answers. A large number of pupils however gave the answer as $22\frac{2}{3}$. These pupils thus gave a numerically correct answer but did not take into account that *people* were being transported! The same phenomenon was found in other items as well. In one item pupils had to calculate an athlete's speed from a quadratic equation. The solutions were 65 and 25 km per hour. Most pupils stuck to both answers without thinking that nobody can run at 65 km per hour.

The low difficulty values of items already discussed under impossible problems and conversions indicate that pupils do not test their solutions for meaningfulness. If only one substitution had been made for numbers leading to simple calculations, the pupils might have seen that their solutions were wrong. The suspicion which can be strongly voiced here is that pupils restrict themselves to their calculations. As soon as an answer is obtained, pupils stick to it. This search for numerical answers appears in many other examples. In problems like $2x - \sqrt{x+4} = 7$ most pupils who gave as their answers $x = \frac{9}{4}$ or $x = 5$ were satisfied with both solutions. (Many pupils make the typical error of writing the next step as $4x^2 - x + 4 = 49$.) This neglect to check the answers was also found in other problems such as solving exponential equations. Mathematics is probably seen by many pupils as the use of techniques and methods which are the result of processes which are not understood, principles which have not been mastered and other such causes.

4.2.4 Letters versus numbers

In the study equivalent items were included to investigate the effect of using letters instead of numbers. The items were equivalent in the sense that the wording was identical but the numbers were replaced by letters. Here is an example:

Item: *If I was 8 years old 12 years ago, how old am I now?*

Equivalent item: *If I were x years old y years ago, how old am I now?*

A total of twelve sets of items of this nature was included in the study. The table below gives the difficulty values of the items in which numbers were used (X) and the equivalent items in which letters were used (Y):

ITEM (X)	M(X)	ITEM (Y)	M(Y)	D	D^2
A	18,4	O	7,6	10,8	116,64
B	22,0	P	22,6	−0,6	0,36
C	61,9	Q	69,9	−8,0	64,00
D	70,1	R	45,8	24,3	590,49
E	49,4	S	34,9	14,5	210,25
F	50,5	T	8,1	42,4	1797,76
G	44,3	U	41,2	3,1	9,61
H	35,5	V	11,6	23,9	571,21
I	57,9	W	20,0	37,9	1436,41
J	64,2	X	14,1	50,1	2510,01
K	87,0	Y	60,8	26,2	686,44
L	88,6	Z	62,6	26,0	676,00

M(X) and M(Y) are the difficulty values of the item and the equivalent item respectively.

The t–test for related data (see for instance Mulder, 1981:146/147) was used to determine whether the observed differences are statistically significant. The marks of a total of 4 635 pupils could be used for this purpose. The t–value was calculated with the formula:

$$t = \frac{\sqrt{N-1}\sum D}{\sqrt{N\sum D^2 - (\sum D)^2}}$$

where

N = the number (rows), and

$D = X - Y$

For $N = 12$, $\sum D = 250,6$ and $\sum D^2 = 8669,18$. A t–value of 4,09 was obtained by substituting these values in the formula. For 11 degrees of freedom $(12 - 1)$ the critical values are 2,201 and 3,106 on the 5 % mark and 1 % mark respectively. Since the values obtained are higher than 3,106 it may be accepted that the observed differences can with 99 % certainty not be put down to chance. The conclusion to be drawn from this is that pupils have more problems with items in which letters are used than with items using numbers.

In the same way items in which the letters x and y were used were played off against equivalent items in which other letters were used. The author suspected that pupils were accustomed to the use of the letters x and y in Mathematics and might experience problems if other letters were used. A difference was found in the five pairs of items included for this purpose (pupils scored higher marks on the items with x and y) but the difference was not significant — a t–value of 0,258 was obtained which is lower than the critical value of 2,776 on the 5 % mark.

4.3 BASIC CONCEPTS AND ACHIEVEMENT IN MATHEMATICS

A total score was obtained for the candidates' achievements in the test with a reliability of 0,950 (K–R 14) and 0,949 (K–R 8). The candidates' average Mathematics mark at the end of Standard Eight was also obtained in most cases. For an indication of the relationship or lack thereof between the two sets of marks the correlation between them can be calculated. The correlation is expressed in terms of an index, the correlation coefficient, which can vary between –1 and 1. The further the index is from nought the greater is the correlation. A positive index implies a direct connection, whilst a negative index indicates an indirect connection.

Pearson's product moment correlation coefficient, r, was calculated with the aid of a computer according to the formula:

$$r = \frac{N\sum xy - \sum x \sum y}{\sqrt{[N\sum x^2 - (\sum x)^2][N\sum y^2 - (\sum y)^2]}} \qquad (Mulder, 1981 : 73)$$

where

N = number of pupils

x = raw test score

y = Standard Eight Mathematics mark.

An approximate value of 0,619 was obtained for N = 4 462. (173 of the 4 635 pupils were excluded because their Standard Eight Mathematics marks were unavailable.) According to Mulder (1981:74) a correlation above 0,60 may be regarded as a high positive correlation.

The null hypothesis ($H_0 : \rho = 0$) used here states that the mastery of basic mathematical concepts, principles and skills does not predict achievement in Mathematics. The alternative hypothesis is posited that they do predict achievement in Mathematics. The question is thus whether the correlation of 0,619 is significant. This may be determined by putting the null hypothesis $r = 0$ against the alternative hypothesis $r = 0,619$. With the aid of a $z-transformation$ of the sample correlation coefficient (r) an approximate confidence interval for ρ (the population correlation coefficient) was found. For $r = 0,619$ and N = 4 462, $z_r = 0,725$. The standard deviation of this is 0,015. The significance of z_r is 48,398 (Barnard, 1988:133). This value is far greater than the critical value of 2,579 — the critical value on the 1 % level. It may therefore be assumed that the correlation is significant. The lower and upper limits of the correlation coefficient were also calculated and values of 0,601 for the former and 0,638 for the latter were obtained for $\alpha = 0,05$. The range (0,638 - 0,601 = 0,037) is relatively small.

We may conclude from the above that mastery of fundamental mathematical concepts, principles and skills is a relatively good predictor of general achievement in Mathematics. Although a correlation is not necessarily an indication of cause–effect, one can say of the results of the investigation as a whole that pupils who have not mastered the fundamental mathematical concepts, principles and skills generally do badly in Mathematics as a subject.

4.4 SUMMARY AND CONCLUSIONS

It seems impossible to find a set classification and/or hierarchy of errors because on the one hand the same problem can give rise to various types of errors and on the other the same mistake can arise from a variety of situations. It can be said however that errors made in Mathematics cannot in most cases be attributed to unfortunate events — errors are the result of processes that are not understood, principles that haven't been mastered and other related causes. Radatz (1979:170) agrees that " ... errors in the learning of mathematics are not simply the absence

of correct answers or the result of unfortunate accidents. They are the consequence of definite processes whose nature must be discovered".

This investigation concentrated on tracing general tendencies in the handling of fundamental mathematical concepts and principles. An attempt was made to explain general errors made by large numbers of pupils. If 50 % or more of the pupils made the same error, it was regarded as a *typical error*. Typical errors were found in various items on different topics. In the literature we find that the same kinds of errors are also made in other countries. The author believes that these errors are largely attributable to the combined effects of interference, regression, autonomous processes and forgetting because of inadequate mastery of fundamental concepts, principles and skills. These factors may be regarded as primary factors in causing the errors that pupils make. If the formation of concepts in the pupils' cognitive structures is superficial or incomplete, the pupils show their inability to handle assumed knowledge very clearly in their confusion of concepts and failure to achieve. Concepts which may have been understood up to a point at first are later easily misapplied. This is one of the main reasons for such mistakes as $0,3 + 0,4 = 0,07$. Pupils learn that $0,3 \times 0,4 = 0,12$ but they do not really understand the concept of place values. This superficial control leads pupils to make their own rules. This invention of rules often appears when pupils cannot remember the method they have memorized. They use what they can remember combined with the easy way out to make their own rules, for instance $2x + 3x = 5x^2$. This example shows a definite confusion of $2x + 3x$ with $2x \times 3x$. There are many other examples of this kind of thing, for instance $\frac{2}{3} + \frac{1}{7} = \frac{3}{10}$ or $(x + 3)(x - 4) = 5 \Rightarrow x + 3 = 5$ or $x - 4 = 5$ based on the zero product principle. Pupils could probably give the correct answer while they were being taught fractions but once multiplication of fractions was explained they tended to make this typical error because on the one hand the concepts had not been properly mastered and on the other it is easier to add numerators to numerators and denominators to denominators. (Multiplication of fractions is easier to remember and so they invent their own rule for addition.) No reference was made to the connections between the calculations. Similarly, few pupils know that a square is a special rectangle or that x^2 is to be found in the tables memorized in primary school - $1 \times 3 = 3; 2 \times 3 = 6; 3 \times 3 = 9; \ldots$ In all the tables there is a case where both numbers are the same and this operation is written in a different way (exponential notation). Venter (1978:49) has this to say: "A specific topic is broken down into a succession of stimulus-and-response situations. All the pupil has to do is give a mechanical response to every stimulus without asking himself whether the specific response is a logical and meaningful one

which fits into the structure of the whole. In this way the child's abilities are not extended and his cognitive development is limited. His unity of vision is lost and this leads to compartmentalized knowledge."

Pupils often get derailed if several steps are required in calculations. They concentrate on one process and make careless mistakes probably through lapses of concentration. Regression comes into the picture as the items get more difficult. Pupils lose their insight into the situation and their thinking drops to an autonomous level.

Following Davis (1984) we can use the following explanation as an illustration: In the first school years children learn to add two single digit numbers together. Because nothing but addition is taught at this stage the children learn after a number of "sums" to fixate only on the numbers in question (without paying much attention to the + symbol). They do not learn to distinguish between different operational symbols because only one is known to them at this stage. When at a later stage multiplication and tables are learnt the pupils are so used to concentrating on the numbers that they readily make the typical error $4 \times 4 = 8$. They are inclined to add the two numbers because this has become a habit. This argument can be taken a step further. By the time exponents are taught pupils have "gotten used to" multiplying. This accounts for typical errors like $2^3 = 6$. The error in the last example is of course made worse by the misunderstanding of exponential notation. Pupils do not appear to know that in $2 + 3$ the 3 stands for a group or set of 3 elements; in 2×3 the 3 stands for the number of groups of two which must be combined; and in 2^3 the 3 stands for the number of times the twos must be multiplied by themselves. Typical examples like $8 \div \frac{1}{2} = 4$ can also be explained in this way. Pupils are taught that half of a number means that the number is divided by two. This is true for the purposes it is used for, but no explicit attention is given to the operational symbol in question. These kinds of errors are largely explicable in terms of the pupils' tendency to apply what has been drilled into them without understanding what they are doing.

Pupils also mostly believe that multiplication always has to lead to a larger answer and so they give the answer to a problem like $6 \times \frac{1}{2}$ *as twelve. Similarly they often believe that division always has to lead to a smaller answer; this explains the answer of 4 to the question* $8 \div \frac{1}{2} = \ldots$

Page (Davis, 1984:100/101) believes that pupils repeatedly give the correct answers to questions that were not asked. During remediation he recommends that teachers should ask the question that the pupils have answered, for instance:

Teacher: How much is 7×7?

Pupil: 14

Teacher: How much is $7 + 7$?

The child will then usually give the correct answer to the first question.

It would appear that pupils use many of the algorithms and methods without really understanding the underlying processes and principles. They often reach correct answers without grasping what they are actually doing. Several authors have cautioned about this. Venter (1978:23/24) says for instance that: "The learning of a recipe without understanding taking place prevents and hinders the establishment of new or adapted thought structures and gives rise to the complaint that the subject is not understood." Venter (1978:60) points out that the initial results of learning recipe techniques parrot-fashion without insight seem to be very good but turn out to be unsuccessful in the long run. De Wet (1966:267) emphasizes this point by saying that: "The meaning and understanding of the working techniques are not stressed. Presenting the material in this way gives rise to a too early lapse into mechanical skills ... " Finally Moodley (1987:4) says that: "Efficient teaching which may often lead children step by step to follow recipes in mathematics can easily be mistaken for good teaching."

The above may be ascribed to two main causes. Firstly it is possible that pupils are expected to handle certain concepts too early. Du Toit (1975:84) points out for instance that the ability to add and subtract develops far later than people think. Pupils are actually forced to memorize the combinations without understanding. Just as it would be untrue to say that a child of two who can count has understood numbers, so it would also be untrue to say that a pupil who performs operations without understanding has insight into what he is doing. The second cause is closely related to the first. The author suspects that teachers for various reasons teach children a method instead of a principle or concept. Understanding is equated with algorithms. Instead of children being told, for instance, why fractions with different denominators cannot be added together, they are taught an algorithm for getting the right answer. In most cases only one traditional method is taught. According to Plunkett (1981:178) the reason for this lies in the nature of the algorithms taught:

1. They are in written form so that the steps can be revised.

2. They are standardized in that the same steps are followed.

3. They are compact in that they summarize various rules of distributivity and associativity.

4. They are effective. For addition it would be less effective to add the tens first because once the units have been added an adjustment might have to be made.

5. They are automatic. **They can be learnt and carried out by people who have no idea of what is happening.**

6. They are symbolic. Calculations can be performed without reference to the real world.

7. They are universal in that they apply to any kind of number work, large or small, whole numbers or decimals, with few or many digits.

8. They are analytical. Numbers are broken up into tens, units and so on and the figures are then manipulated separately.

9. They are not readily internalized, in other words they do not correspond with the ways in which people are inclined to think about numbers.

10. They encourage cognitive passivity. No thought is required about what method should be used or why certain steps are performed in certain ways.

Plunkett (1981:179) points out that even if all the rules can be recalled they are mostly learnt without reasons. Memorization takes place without any reference to other knowledge structures. This is what causes pupils to believe that Mathematics is essentially arbitrary and only involves the application of certain methods and algorithms. This practice is very widespread. For sketching parabolas, for instance, pupils often memorize a formula for obtaining the axis of symmetry rather than simply working out the arithmetical mean of the two roots. The latter is far easier, involves no memorizing, and gives pupils a far better understanding of what the concept of symmetry actually involves.

Teachers make an important contribution to the pupils' belief that Mathematics is simply the application of methods and algorithms. De Wet (1966:103/104) says for instance that: "This is clear from the language in which the statements and operational techniques is given, for instance that one may divide the numerator and the denominator by the same amount or that one must do such and such." This kind of language tends not to be accompanied by explanations of why it

is so. Pupils then accept the statements without understanding the underlying reasons. It is impossible to practise Mathematics simply by memorizing a number of rules and algorithms. Davis (1984:15) is right in saying: "If there were only a few mathematical techniques that needed to be learned, one might consider the possibility of the rote teaching of routine methods. This is not the case. The number of mathematical techniques that may be needed is exceedingly large, far too large to be taught in their entirety."

Pupils are inclined to take a straight jump from Arithmetic to Algebra. It is true that arithmetical working takes place for every algebraic working, but interference cannot follow from this without further ado. It is essential for algebraic operations that pupils should develop manipulative skills in respect of a wide variety of operations, in other words pupils must be able to perform the operations and especially understand the principle on which they are based. Pupils do not however distinguish between algebraic and arithmetic addition, for instance. In the primary school addition means the finding of the cardinal number of two or more sets which are added together. It is almost like a kind of machine with numbers fed in on one side and answers emerging from the other. In Algebra addition, $x + y$, say, means the sum of whatever constants are to be written instead of x and y. The difference is clear once negative numbers come into the picture. Algebraically speaking $3 + (-4)$ is addition but arithmetically speaking it is subtraction. Both addition and subtraction (in the arithmetical context) fall under algebraic addition. Pupils do not however make this distinction.

Küchemann (Booth, 1984:4/5) classifies the use of letters in Algebra into six categories:

i. Letters are evaluated, in other words numerical values are attached to letters.

ii. Letters are ignored.

iii. Letters are treated as objects. Letters are seen as symbols which represent objects or as the objects themselves.

iv. Letters are seen as specific but unknown numbers with which calculations can be performed.

v. Letters are seen as generalized numbers so that different numbers are associated with the same letter.

vi. Letters are seen as variables. A letter is seen as the representation of a series of unspecified values combined with a systematic relationship between two such sets of values.

The abstract nature of the symbolism of Mathematics is the cause of many errors made because of the confusing use of unknowns (as function, as variable and as entity), and so on. Letters are not interpreted by pupils in the context in which they are used. An error like $2x + 3y = 5xy$ reflects the pupil's incorrect interpretation of the letters being used. This pupil regards the letters x and y as objects which can be added together with no trouble. On the other hand a pupil who believes that $x + y$ can never equal $x + k$ sees the letters as specific unknowns - since the letters y and k are different the equation can never be satisfied. It seems that pupils do not realize that the value of the unknown is independent of the letter being used, but think that if the letter is changed the value also changes. (Compare Booth, 1984:5.) Pupils interpret letters in different ways and the most widespread interpretation of letters is that they represent specific but unknown numbers. It is clear from this and other reasons why pupils experience so many problems with the letters in Algebra. Pupils often confuse the roles of the letters in an equation — in $y = mx + c$ some pupils do not know that the x and y are variables while the m and c are placeholders for certain specific values of numbers for the line in question. Because of this pupils are inclined to write down any values in the place of any letters.

Associated with this was the finding that mathematical language poses problems to many pupils. They experience it as a content–free language full of hollow phrases. Because they do not understand the language they cannot express themselves in it. Gouws (1968:2) reaches the same conclusion: "Word determination is neglected and pupils do not understand the words in use. The example can only be understood if the terms and notations are understood and put into the correct context." We have previously mentioned that pupils do not always put the words and symbols into the correct context. The expression $2xy$ will have a value of 24 in an algebraic context if $x = 3$ and $y = 4$ but can be interpreted as 234 in an arithmetical context. The meaning allotted to every figure in a number in Arithmetic depends entirely on the place which it occupies in the number. In an arithmetical context 24 implies an invisible power of ten — $24 = 2 \times 10 + 4$. Pupils' confusion is understandable if one considers that $2\frac{3}{8}$ implies addition $(2 + \frac{3}{8})$. In an algebraic context xy usually indicates multiplication. The invariance of meanings is confusing.

Several examples of language-induced errors showed up in the investigation. Errors like $3x - x = 3$ are attributable to this — a pupil reads *I have 3x and take x away.*

Therefore I have 3 left. Another example which can be mentioned in this context is the classic error of interpretation made with for instance $x(y + z)$. Pupils are inclined to give the answer as $xy + z$. The reason for this is that they probably read *The bracket means multiply. The x must therefore be multiplied by the y because there is a bracket between them. Now the bracket has been used up and the x and the z are separated by just an addition sign. The answer is therefore $xy + z$ because there is nothing outside the second bracket with which the z must be multiplied.*

Together with the problems which may be associated with language we can also mention that it was found in many cases that pupils do not read the questions properly. Face value seems to play an important role. Pupils read into the question what they want to see — perhaps another version of an example they can remember. They look at the numbers in the problem and then search for a word or symbol which suggests that kind of operation. The memorized algorithms are then called up and applied mechanically. Recipe methods are followed slavishly and without insight.

Their pursuit of the correct answer may be why pupils attack problems without thinking about them. This observation is supported by other writers including Moodley (1987:3). He mentions that an analysis of the 1983 "Maths Five Contest" answer sets revealed that some 95 % of the candidates failed to analyse the problems before trying to solve them. The problems were immediately divided up into sections/steps and solved. Moodley refers to the following two examples of problems given:

$\frac{427}{7} + \frac{6 \times 427}{7} = \ldots$ The first fraction was simplified and in the second one the product of the two numbers in the numerator was calculated before the fraction was simplified. Then the two answers were added together.

$1276 \times 48 - 38 \times 1276$. Pupils who got the right answer first worked out the two products and then added the two answers together.

These and other examples show clearly that pupils neglect to analyse the problem before they tackle it. Both of these problems could have been solved in their heads if they had seen that simplification was possible and that calculations were totally unnecessary. Algorithms and methods are applied parrot-fashion to the given numbers. Pupils should rather try to make approximations of the answer or of the order of the answer where possible. If the sum of say $1\frac{3}{5}, \frac{9}{10}, 4\frac{1}{20}$ and $2\frac{17}{30}$ is to be calculated, they could argue that $1\frac{3}{5} \approx 1\frac{1}{2}, \frac{9}{10} \approx 1, 4\frac{1}{20} \approx 4$ and $2\frac{17}{30} \approx 2\frac{1}{2}$

so that the answer should be round-about 9. Pupils should also learn to think beforehand about what kinds of answers will not be acceptable, for instance if a quadratic equation has to be solved to calculate area then a negative answer will not be acceptable even if the algebraic calculations give it as a possible solution.

To summarize one can say that pupils do not always understand what they are doing. Certain rules, methods and algorithms are followed blindly. In many cases pupils do not even know the basic principles on which calculations are based, for instance that if one factor of a product is multiplied by a number, the product is multiplied by that number. This phenomenon shows up in problems like *Calculate the product of 3 and xy*. Pupils are inclined to multiply the x and the y by the 3. One suspects that this method can largely be traced back to the classroom, where too much "spoonfeeding" probably takes place because of overlong syllabuses and so on. This teaching method suppresses divergent thinking on the part of the pupils in order to make place for convergent thinking. Pupils do not learn how to solve problems for themselves but they practise techniques, methods and algorithms according to examples and on the teacher's authority.

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CHAPTER 5
SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 SUMMARY

The purpose of this study was to determine the nature and extent of the defective knowledge of basic mathematical concepts and principles as well as the effect this has on the achievements in Mathematics of pupils in the senior secondary school phase. An attempt was made to form a picture of pupils' knowledge of and skill at handling mathematical concepts and principles. A special effort was made to find possible explanations for errors which occurred frequently. A diagnostic test was developed as a measuring instrument for obtaining interpretable data on the achievement level of Standard Nine pupils in the RSA taking Mathematics as a subject. A total of 4 635 pupils from 106 schools were involved in the investigation as a representative random sample.

The empirical component of the study was based on a theoretical foundation and a thorough study of the literature discussing aspects of Mathematics and the formation of mathematical concepts.

The results of the test, with a reliability index of 0,950 (KR 14), showed that pupils in general revealed an inability to handle mathematical contents with understanding. The average achievement of some 39 % for the test, in which the questions measured the handling of basic concepts and principles as well as skills, gives a direct indication of poorly developed concepts and their handling.

No attempt was made to find a set classification and/or hierarchy of errors because on the one hand the same problem can give rise to various types of errors and on the other the same mistake can arise from a variety of situations. The analysis of the results was divided into three main categories, mathematization, the use of fundamental concepts and principles, and demathematization. The first category revealed problems connected with aspects of language while the third gave an indication of pupils' interpretations of the answers they obtained. The second category was subdivided into sixteen topics which were used to trace tendencies in the erroneous handling of the fundamentals of mathematical practice. Facts were coordinated throughout rather than just being presented as a collection.

5.2. CONCLUSIONS

The study revealed that errors made by pupils in Mathematics are in most cases not simply attributable to accidental causes. Errors are the consequence of concepts, terms and processes not being understood, principles not being mastered and other causes. Initially it may appear that pupils have mastered certain concepts while these are being taught in isolation from other concepts because the problems are solved with the aid of the examples given. As soon as concepts have to be manipulated outside their immediate field, many pupils reveal their inability to handle assumed knowledge.

The incomplete and/or superficial formation of concepts in the cognitive structure leads to the confusion of concepts and weak results. Operations and manipulations are performed mechanically with the aid of rote knowledge, recipes and examples. If a method cannot be recalled, pupils often invent their own rules by taking the easy way out. These rules can in most cases be traced back to processes more or less mastered earlier and then incorrectly applied out of context. The example of a direct jump from Arithmetic to Algebra may be mentioned here. Many pupils' ability to use even fundamental processes in Algebra is relatively poorly developed and so they resort to Arithmetic if they cannot recall the techniques and methods which they had not properly mastered. Interference, regression and autonomous processes combined with forgetfulness are thus regarded as primary causes of errors.

Two main causes singled out as reasons for making errors and handling concepts poorly are that contents are presented in isolation from each other and that methods and algorithms are learnt instead of concepts. The author believes that many teachers present a method for adding fractions for instance instead of first ensuring that pupils have intuitively grasped the concept of fractions. Pupils then get drilled once the method has been memorized. This may explain why pupils can do workings with simple fractions but fail as soon as the concept of fractions has to be handled. This early mechanization leads on the one hand to initial success but subsequent failure and on the other to mindless shifts and manipulations of numbers, letters and mathematical symbols. It does not mean a lot for instance if a pupil tries to learn the meaning of $x^m.x^n$ by keeping on changing the values of m and n in a number of examples if the concept has not been internalized. Instead of training pupils to perform certain manipulations, the emphasis in teaching should be on giving meaning to the basic principles and concepts of Mathematics. Pupils should for instance first understand commutative, associative, distributive, identity and inverse properties before rules like: the sign of the product of two negative numbers

is positive; if the product of two numbers is zero then at least one of the numbers must be zero; to add like terms the coefficients of the terms may be added together; and so on can mean anything to them at all.

Teachers cannot simply accept that contents learnt in previous standards and concepts which have to be built on have been mastered as one would wish. This would mean that the contents and concepts will become progressively more obscure to the pupils. Teachers should meet their pupils where they are, and not where they presumably should be. In particular it is of cardinal importance to establish whether the basic concepts and principles fundamental to Mathematics have been mastered. Even if syllabuses change, these aspects will continue to be decisive for progress in Mathematics.

We may point out in conclusion that concepts first have to be internalized in concrete form before they can be abstracted. The meaningfulness of teaching fractions using pictures in a textbook is for instance questionable unless the children have physically played with parts of a whole. It is also vital to mention that a child needs to be intellectually and also emotionally mature to be receptive to the learning of concepts on all levels.

5.3 RECOMMENDATIONS

Various possibilities and perspectives for future research in this area have been opened up by this investigation. Similar umbrella studies could be undertaken in Geometry and Trigonometry, or investigations could involve other standards. A similar investigation in Standard Seven might have great practical value in view of the importance of subject choices at this level. Intensive studies on each of the investigated topics, or on other topics, possibly supplemented by interviews, smaller investigations with open items and so on could lead to valuable contributions.

The value of educational research lies particularly in its practical implications, and so three concluding thoughts will be presented in that context.

(i) Mathematics must never be taught in isolation from reality. A problem which in my view has not yet been satisfactorily answered is why pupils cannot solve a particular problem in the classroom but are able to solve it once it is not written in abstract symbols. (Pupils can for instance divide sweets evenly among themselves but as soon as the problem is posed mathematically - dressed up in terminology and symbols - they fail to find the right answer.) This problem could be scientifically

approached by perhaps searching for the discrepancy in the movement from concrete to abstract.

(ii) Personal teaching experience has shown that the application of diagnostic tests is a laborious task. The tests are usually long in an attempt to ensure completeness. Even if the marking procedures are designed for quick completion they still do take time. If one also bears in mind that diagnostic tests are usually applied individually, the extent of their application and correction becomes clear. In view of technological progress and the reasonable availability of microcomputers one wonders whether this kind of test should not be computerized on a large scale. One of the most attractive advantages of this, apart from the fact that very little of the teacher's time would be taken up by the administering of the tests, is that immediate feedback could be built into the programme. If a pupil gives a wrong answer to a question, the correct answer together with its workings and so on could instantly appear on the screen. In other words the diagnosis would be immediately succeeded by remediation. Research in this regard could be very valuable.

(iii) On the basis of the results of this study, one can reasonably consider the desirability of teaching Mathematics in terms of mastery learning. Surely the hierarchical nature of the subject makes it vital that it should be subdivided into units so that a lower unit is first mastered as it should be before moving on to the next unit? The time factor becomes less important and the mastery factor more important. More research into this area is essential for South Africa.

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